On A Sign Controller for the Triple Integrator
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Abstract—It is well known that for a double integrator a controller using only the signs of the state variables, the Twisting controller, is able to bring all trajectories to the origin in finite time and robustly with respect to bounded matched perturbations. In this paper we analyze a triple integrator with a controller consisting of the sum of the signs of the state variables multiplied by constant gains. We show that, in contrast to the twisting controller, there is no set of gains for which the origin is a globally asymptotically stable equilibrium point. There is however a set of gains such that for almost all initial conditions the trajectories converge in finite time to the origin.

I. INTRODUCTION

Sliding Mode Control (SMC) is well known for its insensitivity to matched perturbations. This strong robustness feature derives from the use of a discontinuous controller: the sign function. In the simplest case of a first order plant (or a system with relative degree one), the SM controller is given by the discontinuous feedback of the state:

\[
\dot{x} = u + \delta(t), \quad u = -k_1 \text{sign}(x).
\]

All trajectories of (1) are driven to the origin \(x = 0\) in finite time and kept there for all future times if the perturbation term \(\delta(t)\) is bounded, i.e. \(|\delta(t)| \leq \Delta\) for all \(t \geq 0\) and \(\Delta < k_1\). This is the “magic” of the sign function (see for example [9]).

For the double integrator

\[
\dot{x}_1 = x_2, \quad \dot{x}_2 = u + \delta(t),
\]

the “magic” of the sign is repeated by the well-known Twisting algorithm [6]: the simple feedback of the sign of each state variable, i.e.

\[
u = -k_1 \text{sign}(x_1) - k_2 \text{sign}(x_2)
\]

is able to drive in finite time the trajectories to the origin \((x_1, x_2) = 0\) and to maintain them there whenever the perturbation \(\delta(t)\) is bounded (by \(\Delta\)) and \(k_1 - \Delta > k_2 > 0\). Besides their robustness properties these controllers are very appealing due to their simplicity, since only the sign of the states is feedback.

In [1] the triple integrator:

\[
\dot{x}_1 = x_2, \quad \dot{x}_2 = x_3, \quad \dot{x}_3 = u
\]

is controlled using only the sign of the states, but the algorithm consists of two different control schemes that have to be switched with certain logic. The switching criteria induces a discrete state [7]. This kind of controller was generalized to arbitrary order in [5].

With the aim of stabilizing (3) is natural to think in the controller:

\[
u = -k_1 \text{sign}(x_1) - k_2 \text{sign}(x_2) - k_3 \text{sign}(x_3)
\]

that is the extension of the algorithms (1) and (2) to order three. The closed loop (3), (4) has already been treated in [8]. There it is said that the origin of (3) can be finite-time stabilized with the controller (4) for certain set of gains. The purpose of the present paper is to show that the origin of (3) cannot be globally-asymptotically stabilized with the controller (4) for any set of gains. However it will be proved that there exists a set of gains such that “almost all” trajectories of the closed loop (3), (4) are driven to the origin in finite time. The remaining trajectories converge in finite time to a set of equilibrium points out of the origin.

In Section two the main result is presented and discussed. The proof of the result is given in Section three (due to its length it is not presented in detail) and finally some conclusions are given in Section four.

II. THE MAIN RESULT

Consider the third order dynamical system (3) with state vector \(x = [x_1, x_2, x_3]^T \in \mathbb{R}^3\) and with control \(u \in \mathbb{R}\) is given by (4). Solutions of the closed loop (3-4) are understood in the sense of Filippov [2].

The controller (4) is piece-wise constant, this means that:

\[
u = \begin{cases} u_1 = k_1 + k_2 - k_3, & x_1, x_2 < 0, x_3 > 0 \\ u_2 = -k_1 + k_2 + k_3, & x_1 > 0, x_2, x_3 < 0 \\ u_3 = k_1 + k_2 + k_3, & x_1, x_2, x_3 < 0 \\ u_4 = k_1 - k_2 + k_3, & x_1, x_3 < 0, x_2 > 0 \\ u_5 = -k_1 - k_2 + k_3, & x_1, x_2 > 0, x_3 < 0 \\ u_6 = k_1 - k_2 - k_3, & x_1 < 0, x_2, x_3 > 0 \\ u_7 = -k_1 - k_2 - k_3, & x_1, x_2, x_3 > 0 \\ u_8 = k_1 + k_2 - k_3, & x_1, x_3 > 0, x_2 < 0, \\
\end{cases}
\]

where

\[
u_5 = -u_1, \quad u_6 = -u_2, \quad u_7 = -u_3 \quad \text{and} \quad u_8 = -u_4.
\]

We will suppose that the gains take only positive values, this is \(0 < k_1, k_2, k_3 \in \mathbb{R}\). It follows that \(u_3 = -u_2 > 0\).

To state the main result in Theorem 2.1 we need the following definitions:

- The set \(\mathcal{K} \subset \mathbb{R}_+^3\) is a subset of the gain space such that

\[
\mathcal{K} = \{ k \in \mathbb{R}_+^3 | k_1 < k_2 + k_3, \quad k_2 < k_1 + k_3, \quad k_3 < k_1 + k_2, \quad k_3 > k_1 \}.
\]
• For every \((k_1, k_2, k_3) \in K\) the set \(S \subset \mathbb{R}^3\) is a surface in the phase space (see Figure 1) defined as
\[
S = \{x \in \mathbb{R}^3 \mid x_1 x_2 \leq 0, \ x_2 x_3 \leq 0, \ x_3^2 = 2u_4|x_2|, \ x_2 x_3 > -3u_4|x_1|\}.
\]
• For each point \(x \in S\) we define a corresponding point \(E(x) \in S\) as follows:
\[
E(x) = \begin{bmatrix} x_1 + \frac{\text{sign}(x_1)}{3u_4} x_2 x_3, \ 0, \ 0 \end{bmatrix}^T.
\]

**Theorem 2.1:** Consider the closed loop system (3-4).
1) The origin \(x = 0\) is not globally asymptotically stable for any set of gains \(k_1, k_2\) and \(k_3\).
2) Suppose that \((k_1, k_2, k_3) \in K\). In this case:
   a) The set \(\mathbb{R}^3 \setminus S\) is invariant, i.e. every trajectory starting in it remains in the set for all times.
   b) Every trajectory with initial condition in \(\mathbb{R}^3 \setminus S\) converges in finite time to the origin.
   c) The set \(S\) is invariant.
   d) Every trajectory with initial condition \(x_0 \in S\) converges in finite time to the point \(E(x_0)\), which is an equilibrium point.

The Theorem is proved in Section III. First it is shown that for all gains there exist trajectories that are not attracted to the origin. Then, by restricting the gains to the set \(K\), we show that the set \(S\) and its complement \(\mathbb{R}^3 \setminus S\) are invariant. To prove item 2b) we construct a Lyapunov function defined in \(\mathbb{R}^3 \setminus S\). The proposed Lyapunov function \(T(x)\) corresponds to the reaching time to the origin of the trajectory starting at the point \(x\), and it is given in Section III-D, and the Appendix. Finally item 2d) is proved by a direct calculation. In the next paragraphs we want to make a brief discussion and illustrate some of the results of the theorem, in particular item 2).

**Example 1:** This example illustrates the behavior of the trajectories that initiate in \(\mathbb{R}^3 \setminus S\) when the control gains are chosen from the set \(K\). Consider (3-4) with gains \((k_1, k_2, k_3) = (2, 2, 2.5)\). In Figure 2 it can be seen that the states converge to zero in finite time for two initial conditions \(x_0^T = [1, -7, -2]\) and \(x_0^T = [-6, 2, 5]\). Figure 3 shows a trajectory in the phase space. Note that it undergoes a First Order Sliding Mode in some regions of the plane \(\{x_3 = 0\}\).

**Example 2:** We illustrate the behavior of the trajectories on \(S\), that do not converge to the origin. Set the gains (in \(K\)) as in the previous example and the initial condition \(x_0^T = [-5, 5, -5]\) \(\in S\). Figure 4 shows that the trajectory converges in finite time to the final value \(E(x_0) = [5/3, 0, 0]^T\).

Note that \(S\) is a measure-zero set in the state space, and it can be proved (Section III) that it is also an invariant set. Therefore it could be thought that the set \(S\) is not quite “problematic” (for control purposes) because every trajectory whose initial condition is out of \(S\) will converge to the origin in finite time. But let us show, by mean of an example, the situation of the trajectories that start near the surface \(S\).

**Example 3:** Again consider (3), (4) with the gains as in the Example 1. Now fix the initial conditions \(x_{10} = -5\) and
Fig. 4. States of (3), (4) when $x_0 \in \mathcal{S}$ and the controller’s gains in $\mathcal{K}$.

$x_{20} = 5$. In Figure 5 is shown the following:

1) If $x_{30} = -5.5$ states go to zero in approximately twenty units of time
2) In the middle plot $x_{30} = -5.1$ and the states go to zero in a little bit more than forty units of time
3) Finally when $x_{30} = -5.05$ the states go to zero in about ninety units of time

Clearly the convergence time to the origin is larger for initial conditions closest to the set $\mathcal{S}$. So that not only $S$ is a “problematic” set but also its vicinity. The same phenomenon can be appreciated in the Figure 6 where $\mathcal{T}(x) \to \infty$ as $x \to \mathcal{S}$.

III. PROOF OF THEOREM 2.1

The proof of the first statement of the Theorem 2.1 will consist in analyzing the trajectories behavior for the whole set of gains. The set of gains will be divided in four subsets.

The analysis for the last subset will include the proof of the second statement of the Theorem.

A. Analysis for the subset $\{k_1 \geq k_2 + k_3\}$

It is going to be proved that the origin of (3), (4) is unstable when the controller’s gains are chosen from $\{k_1 \geq k_2 + k_3\}$. The proof will be done through the Instability Chetaev’s Theorem (see for example [4]). Consider the following function (this function was taken from page 66 of [3]):

$$V = -x_1x_3 + \frac{1}{2}x_2^2,$$

in particular this function is positive over the set $\{x_1 > 0, x_2, x_3 < 0, x_2^2 > 2x_1x_3\}$. If we take its derivative along the trajectories of (3), (4) we have that

$$\dot{V} = -x_1[-k_1 \text{sign}(x_1) - k_2 \text{sign}(x_2) - k_3 \text{sign}(x_3)]$$

$$= k_1|x_1| - x_1[-k_2 \text{sign}(x_2) - k_3 \text{sign}(x_3)]$$

$$= k_1|x_1| - x_1[k_2 + k_3]$$

$$= |x_1|(k_1 - k_2 - k_3).$$

Thus $\dot{V} > 0$ when $k_1 > k_2 + k_3$ and the origin unstable. If $k_1 = k_2 + k_3$ the origin is not attractive.

B. Analysis for the subset $\{k_3 \geq k_1 + k_2\}$

First look from (3) that in the set $\{x_1, x_2 > 0, x_3 < 0\}$ the vector field of the phase space is $[x_2, x_3, u_3]^T$. Now suppose that $u_3 \geq 0$ which is equivalent to $k_3 \geq k_1 + k_2$. Then, in the mentioned set, for a sufficiently large initial condition $x_2(k_0)$ there will exist trajectories that will be driven to the plane $\{x_3 = 0\}$. On the other hand, in the set $\{x_1, x_2, x_3 > 0\}$ the control input is $u(x) = u_7 < 0$ and the vector field is $[x_2, x_3, u_7]^T$ which points toward the plane $x_3 = 0$. Therefore a first order Sliding Mode behavior is going to take place in a subset of $\{x_3 = 0\}$ in this sliding surface we have the reduced system

$$\dot{x} = x_2, \quad \ddot{x} = 0$$

from which it is clear that $x_2$ will establish in a positive value and $x_1$ will grow indefinitely. So that, in this way we have concluded instability of the system’s origin when the algorithm’s gains are chosen such that $k_3 \geq k_1 + k_2$. 

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C. Analysis for the subset \( \{k_2 > k_1 + k_3\} \)

Consider from (3), (4) the two order subsystem
\[
\dot{x}_2 = x_3, \quad \dot{x}_3 = u
\]
that can be rewritten as follows:
\[
\dot{x} = x_3 = f(x_1) + v
\]
where \( f(x_1) = -k_1 \text{sign}(x_1) \) and \( v = -k_2 \text{sign}(x_2) - k_3 \text{sign}(x_3). \) Note that \( |f(x_1)| \leq k_1. \) Thus (5) can be seen as a Twisting algorithm [6] with \( f(x_1) \) as a perturbation input. According to [6] the following are the two conditions that guarantee origin’s finite time stability of (5)
\[
k_2 - k_3 > k_1 \text{ and } k_2 + k_3 - k_1 > k_2 - k_3 + k_1,
\]
note that these two conditions are equivalent to
\[
k_2 > k_1 + k_3 \text{ and } k_3 < k_2 + k_3.
\]

In the best case the last two inequalities are fulfilled, then in (3), (4) the states \( x_2 \) and \( x_3 \) will be annulled in finite time regardless the value of \( x_1 \). Thus, due to \( \dot{x}_1 = x_2, \) \( x_1 \) will settle down in a constant value in general different to zero. Hence we have proved that the condition \( k_2 > k_1 + k_3 \) cannot guarantee the global-asymptotic origin’s stability.

D. The remaining gains

In this Section we suppose that the inequalities \( k_1 < k_2 + k_3, k_2 < k_1 + k_3 \) and \( k_3 < k_1 + k_2 \) hold. The aim of this Section is to complete the proof of the first statement of Theorem 2.1 and also to proof the second one. To this end we are going to try to design a Lyapunov function for the system (3), (4). We will use the method proposed in [7], this method is as follows:

Let \( \dot{x}(\tau; 0, x) \) denote the solution of (3), (4) with initial conditions \( (t, x) = (0, x) \). Then \( T : \mathbb{R}^n \rightarrow \mathbb{R} \) with
\[
T(x) = \int_0^\infty W(\dot{x}(\tau; 0, x)) d\tau
\]
is a Lyapunov function for (3), (4) if and only if the integral in (6) converges for some positive definite function \( W(x) \). Furthermore \( T = -W(x) \). If the function \( W(x) \) is chosen as \( W(x) = 1 \) then (6) must be replaced with the following equation:
\[
T(x) = \sum_{i=1}^\infty T_i,
\]
where \( T_i \) is each one of the transient times of \( \dot{x}(\tau; 0, x) \) in each subset where the control takes a different constant value.

On the other hand, observe that (3), (4) can be seen as a piece-wise affine system, namely:
\[
\dot{x} = Ax + Bu_m, \quad x \in D_m, \quad m = 1, 2, \ldots, 8
\]
where \( D_m \subset \mathbb{R}^3 \) is the domain where the control \( u_m \) acts and
\[
A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.
\]
Thus, the solution of (3) can be given explicitly by
\[
\begin{align*}
\dot{x}_1 &= \frac{u}{6} x_3 + \frac{u}{2} x_2 + x_2 \tau + x_1 \\
\dot{x}_2 &= \frac{u}{6} x_2 + x_2 \tau + x_2 \\
\dot{x}_3 &= u \tau + x_3
\end{align*}
\]
Note that for any initial condition \((0, x)\) the transient times \( T_i \) can be computed from (8), (9) and (10). Therefore it is possible to use the method aforementioned to try to construct a Lyapunov function for (3), (4).

Using the above method it can be concluded that the function obtained is valid only for the domain \( \mathbb{R}^3 - S \). Thus from (7) we have that the function \( T : \mathbb{R}^3 - S \rightarrow \mathbb{R} \) where
\[
T(x) = \begin{cases} T_1(x), & x \in R_1 \\ \vdots \\ T_{10}(x), & x \in R_{10} \end{cases}
\]
is a Lyapunov function for (3), (4) (restricted to \( \mathbb{R}^3 - S \)) only if \( k_1 < k_3 \). With the aim to clarify the necessity of the restriction \( k_1 < k_3 \) we are going to show the construction of \( T_1 \). Suppose that \( x \) belongs to the set:
\[
R_1 = \{(x_1 > 0, x_2, x_3 < 0) \text{ or } (x_1 < 0, x_2, x_3 > 0)\}
\]
\[\cap \{|x_3|^3 + 3u_2x_2x_3 - 3u_2^2|x_1| < 0\},\]
then from (7) and computing every \( T_i \) we have that
\[
\sum_{i=1}^\infty T_i = \frac{|x_3|}{u_2} - \frac{2}{3u_2} |x_3|^3 + 3u_2x_2x_3 - 3u_2^2|x_1| + \alpha_0 \sqrt{x_3^2 + 2u_2^2} \sum_{i=1}^\infty r^{i-2}
\]
where
\[
r = \sqrt{\frac{u_1}{u_2}} \quad \text{and} \quad \alpha_0 = \frac{1}{u_1} + \frac{1}{u_2} + \frac{2u_2}{3} \left( \frac{1}{u_1^2} + \frac{1}{u_2^2} \right).
\]
Observe in (11) that the term \( \sum_{i=1}^\infty r^{i-2} \) converges to \( \frac{r}{1-r} \) if and only if \( r < 1 \). Thus we have that
\[
r < 1 \iff u_1 < u_2 \iff k_1 < k_3.
\]
Therefore over the region \( R_1 \) the expression for \( T \) is:
\[
T_1(x) = \frac{|x_3|}{u_1} + \alpha_1 \sqrt{x_3^3 + 3u_2x_2x_3 - 3u_2^2|x_1|} - \frac{2}{3u_2} |x_3|^3 + 3u_2x_2x_3 - 3u_2^2|x_1|,
\]
where \( \alpha_1 = \frac{\alpha_0 r}{1-r} \). The condition \( r < 1 \) is also needed in the computation of \( T \) in the remainder space. Note that from \( R_1 \) and (11) can be deduced the existence of the set \( S \). The expressions for \( T_j \) and \( R_j, j = 1, 2, \ldots, 10 \), are given in the Appendix.

According to [7] the derivative along the trajectories of the system (on \( \mathbb{R}^3 - S \)) is \( \dot{T} = -1 \) almost everywhere. Let \( T \) be the convergence time to the origin i.e. \( x(t) = 0, \forall t \geq T \) then
\[
\int_0^T \frac{d}{dt} T(x(t)) \ dt = \int_0^T (-1) \ dt,
\]
thus it is clear that
\[ T(x(T)) - T(x(0)) = -T \iff T = T(x_0). \]
Then the convergence time to the origin \( T \) can be computed exactly. In Figure 6 it can be seen a plot of \( T \) over the set \( \{ x_2 > 0, x_3 < 0, 0 > x_1 = \text{constant} \} \). Note that the convergence time increases for points near to the surface \( S \).

Now we analyze the behavior of (3), (4) over the surface \( S \) when \((k_1, k_2, k_3) \in \mathbb{K}\). Figure 1 shows an example of this surface and its trajectories.

From (9) and (10) it is clear that
\[
\dot{x}_2 = \frac{1}{2u} \left[ (u_1 + x_3)^2 - (x_3^2 - 2ux_2) \right] = \frac{1}{2u} \left[ x_3^2 - (x_3^2 - 2ux_2) \right].
\] (12)

Note that in the region \( \{ x_2, x_3 < 0, x_2x_3 \leq 0 \} \) it holds \( \dot{x}_2 = u_4|x_2| \). Moreover if the initial condition \( x \in S \) (which implies that \( x_3^2 = 2u_4|x_2| \)) then from (12) we get that \( 2u_4|\dot{x}_2| = x_3^2 \). Making a similar analysis with (8) it is easy to get that \( x_2x_3 > -3u_4|x_1| \). Note that if the time is inverted the analysis is valid as well. Therefore it has been proved that \( S \) is an invariant set of (3), (4).

The system’s vector field will generate a second order sliding mode on the set \( \{ x_2 = x_3 = 0 \} \) for all the initial conditions in \( S \). When a trajectory reach this sliding domain it can be seen from (3) that \( x_1 = 0 \). Thus the state \( x_1 \) will establish in a constant value. Therefore \( \{ x_2 = x_3 = 0 \} \) can be seen as an equilibrium set with \( S \) as its attraction domain.

According to (10) a trajectory with initial condition \( x \in S \) will reach \( \{ x_2 = x_3 = 0 \} \) in a time \( \tau_1 = |x_3|/u_1 \). Substituting \( \tau_1 \) in (8) and using the equality \( x_3^2 = 2u_4|x_2| \) we get that
\[ \dot{x}_1(\tau_1) = x_1 + \frac{\text{sign}(x_1)}{3u_4} x_2 x_3. \]

IV. CONCLUSIONS

It was proved in this paper that the origin of the triple integrator (3) cannot be globally and asymptotically stabilized with the Third Order Sliding Mode controller (4). It was also proved that there exists a set of gains for the controller (4) such that for almost every initial condition, the system’s trajectories are driven to the origin in finite time. Although \( S \) is a measure-zero set, it was warned about the increasing of the convergence time for trajectories whose initial condition is near to the set \( S \).

APPENDIX

Over the region:
\[ R_2 = \{ (x_1, x_2 > 0, x_3 < 0) \text{ or } (x_1, x_2 < 0, x_3 > 0) \} \cap \{ |x_3|^3 - 3u_1 x_2 x_3 - 3u_1^2|x_1| < (1 - r^4)(x_3^2 + 2u_1|x_2|)^{3/2} \} \]
the expression for the function \( T \) is:
\[
\mathcal{T}_2(x) = -\frac{|x_3|}{u_1} + \frac{\alpha_2}{3u_2} x_3^2 + 2u_1|x_2| - \frac{2r^4|x_3|^3 - 3u_1 x_2 x_3 - 3u_1^2|x_1|}{3u_2} x_3^2 + 2u_1|x_2|.
\]

where
\[ \alpha_2 = \frac{1}{u_1} + \frac{1}{u_2} - \frac{2(1 - r^4)}{3u_2} + \alpha_1. \]

Over the region:
\[ R_3 = \{ (x_1, x_2, x_3 > 0) \text{ or } (x_1, x_2, x_3 < 0) \} \]
the expression for the function \( T \) is:
\[
\mathcal{T}_3(x) = \frac{|x_3|}{u_3} + \frac{\alpha_2}{3u_3} x_3^2 + 2u_3|x_2| + \frac{2r^6|x_3|^3 + 3u_3x_2x_3 + 3u^2_3|x_1|}{3u_3} x_3^2 + 2u_3|x_2|.
\]

Over the region:
\[ R_4 = \{ (x_1 > 0, x_2, x_3 < 0) \text{ or } (x_1 < 0, x_2, x_3 > 0) \} \cap \{ |x_3|^3 + 3u_2 x_2 x_3 - 3u_2^2|x_1| > 0 \} \]
define
\[ \Gamma = \cos \left[ \frac{\pi}{3} + \frac{1}{3} \arccos \left( \frac{|x_3|^3 + 3u_2 x_2 x_3 - 3u_2^2|x_1|}{(x_3^2 + 2u_2|x_2|)^{3/2}} \right) \right], \]
thus, in this region the expression for the function \( T \) is:
\[
\mathcal{T}_4(x) = \frac{|x_3|}{u_2} - \frac{|x_3|}{u_2} + \mathcal{T}_3(0, x_1, x_31)
\]
where
\[ x_{31} = \text{sign}(x_3)2\Gamma \sqrt{x_3^2 + 2u_2|x_2|} \]
and
\[ x_{21} = \frac{\text{sign}(x_2)}{2u_2} \left[ -x_3^2 + (x_3^2 + 2u_2|x_2|) \right]. \]

Over the region:
\[ R_5 = \{ (x_1, x_2 > 0, x_3 < 0) \text{ or } (x_1, x_2 < 0, x_3 > 0) \} \cap \{ |x_3|^3 - 3u_1 x_2 x_3 - 3u_1^2|x_1| - (1 - r^4)(x_3^2 + 2u_1|x_2|)^{3/2} > 0 \} \]
the expression for the function \( T \) is:
\[
\mathcal{T}_5(x) = -\frac{|x_3|}{u_1} + \frac{|x_3|}{u_1} + \mathcal{T}_4(0, 0, x_{31})
\]
where
\[ x_{11} = \frac{\text{sign}(x_1)}{3u_1^2} \left[ -|x_3|^3 + 3u_1 x_2 x_3 + 3u_1^2|x_1| + (x_3^2 + 2u_1|x_2|)^{3/2} \right] \]
and
\[ x_{31} = \text{sign}(x_3)\sqrt{x_3^2 + 2u_1|x_2|}. \]

Over the region :
\[ R_6 = \{ (x_1, x_3 > 0, x_2 < 0) \text{ or } (x_1, x_3 < 0, x_2 > 0) \} \cap \{ |x_3|^3 + 3u_4 x_2 x_3 + 3u_4^2|x_1| > 0 \}
and \quad x_3^2 + 2u_4|x_2| > 0 \} \]
the expression for the function $T$ is:

$$T_6(x) = \frac{|x_3|}{u_4} + \frac{\alpha_1}{r} \frac{u_{14}}{u_4} \sqrt{x_2^2 + 2u_4|x_2|} + \frac{2}{3u_4} \frac{|x_3|^3 + 3u_4x_2x_3 + 3u_4^2|x_1|}{x_2^2 + 2u_4|x_2|}.$$ 

Trajectories with initial conditions in $R_6$ hit first the plane $\{x_3 = 0\}$. Over the region:

$$R_7 = \{ (x_1, x_3 > 0, x_2 < 0) \text{ or } (x_1, x_3 < 0, x_2 > 0) \} \cap \{ |x_3|^3 + 3u_4x_2x_3 + 3u_4^2|x_1| > (x_2^2 - 2u_4|x_2|)^{3/2} \}$$

and $-x_3^2 + 2u_4|x_2| < 0$.

the expression for the function $T$ is:

$$T_7(x) = \frac{|x_3|}{u_4} + \frac{\alpha_4}{r} \sqrt{x_2^2 - 2u_4|x_2|} + \frac{2u_6}{3u_4} \left( \frac{|x_3|^3 + 3u_4x_2x_3 + 3u_4^2|x_1|}{x_2^2 - 2u_4|x_2|} \right)$$

where

$$\alpha_4 = \frac{1}{u_3} - \frac{1}{u_4} + \frac{2\rho u_3}{3} \left( \frac{1}{u_3^2} - \frac{1}{u_4^2} \right) + \alpha_2 \sqrt{\frac{u_{14}}{u_4}}.$$ 

Over the region:

$$R_8 = \{ (x_1, x_3 > 0, x_2 < 0) \text{ or } (x_1, x_3 < 0, x_2 > 0) \} \cap \{ \Delta < 0 \} \cap R$$

where

$$R = \{ x_2^2 < 2u_4|x_2| \text{ and } |x_3|^3 + 3u_4x_2x_3 < -3u_4^2|x_1| \} \cup \{ x_2^2 > 2u_4|x_2| \text{ and } |x_3|^3 + 3u_4x_2x_3 + 3u_4^2|x_1| > -\left( x_2^2 - 2u_4|x_2| \right)^{3/2} \}$$

and

$$\Delta = \frac{( |x_3|^3 + 3u_4x_2x_3 + 3u_4^2|x_1| )^2 - (x_2^2 - 2u_4|x_2|)^3}{u_4^3}.$$ 

Now define

$$\Gamma = \cos \left[ \frac{\pi}{3} - \frac{1}{3} \arccos \left( \frac{|x_3|^3 + 3u_4x_2x_3 + 3u_4^2|x_1|}{(x_2^2 - 2u_4|x_2|)^{3/2}} \right) \right],$$

thus, in this region the expression for the function $T$ is:

$$T_8(x) = \frac{|x_3|}{u_4} - \frac{|x_3|}{u_4} + T_5(0, x_21, x_31)$$

where

$$x_31 = \text{sign}(x_3)2\Gamma \sqrt{x_2^2 - 2u_4|x_2|}$$

and

$$x_21 = \frac{\text{sign}(x_2)}{2u_4} \left[ x_{31} - (x_2^2 - 2u_4|x_2|) \right].$$

Over the region:

$$R_9 = \{ (x_1, x_3 > 0, x_2 < 0) \text{ or } (x_1, x_3 < 0, x_2 > 0) \} \cap \{ \Delta > 0 \} \cap R \cap R_6$$

where $R$ and $\Delta$ are as before and $R_6$ is a subset of $\mathbb{R}^3$ such that the trajectories of (3) are driven to the plane $\{ x_1 = 0 \}$ and obey the inequality $|x_3|^3 - 3u_1x_2x_3 - (1 - r^4)(x_2^2 + 2u_1|x_2|)^{3/2} < 0$.

Thus, in this region the expression for the function $T$ is:

$$T_9(x) = t_1 + T_5(0, x_21, x_31)$$

where

$$t_1 = \frac{|x_2|}{u_4} + \sqrt{\Delta} + \frac{1}{u_4} \left( (x_3|^3 + 3u_4x_2x_3 + 3u_4^2|x_1|)^{3/2} \right)$$

and

$$x_31 = -\frac{\text{sign}(x_3)}{u_4} t_1 + x_3.$$

Let

$$R_{10} = \{ (x_1, x_3 > 0, x_2 < 0) \text{ or } (x_1, x_3 < 0, x_2 > 0) \} \cap \{ \Delta > 0 \} \cap R \cap R_6$$

be the region where $R$ and $\Delta$ are as before and $R_6 \subset \mathbb{R}^3$ is such that the trajectories of (3) arrive to the plane $\{ x_1 = 0 \}$ holding the inequality $|x_3|^3 - 3u_1x_2x_3 - (1 - r^4)(x_2^2 + 2u_1|x_2|)^{3/2} > 0$.

Thus, in this region the expression for the function $T$ is:

$$T_{10}(x) = t_1 + T_5(0, x_21, x_31)$$

where $t_1$, $x_21$, and $x_31$ are as in the last part.

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