Decentralized $H^2$ Optimal Control of Haptic Interfaces for a Shared Virtual Environment

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Abstract—Decentralized control of haptic interface devices for a shared virtual environment over delayed communication is considered. We show that allowing indirect communication between the interface devices via the virtual environment site may greatly facilitate controller synthesis and analysis by rendering the problem quadratically invariant. In particular, this allows us to find an insightful structure possessed by all stabilizing controllers within the proposed architecture and to show that this structure offers stability of the system regardless of the delay length. We show also that within the proposed control architecture, the global $H^2$ optimization can be reduced to a number of uncoupled model matching problems with multiplicative delays.

I. INTRODUCTION

The notion “virtual environment” (VE) refers to a computer-simulated environment that enables interaction of human operators with virtual objects. During the past decades, the use of VE has been continuously developing in such areas as design [1], manufacturing [2], medical/military training [3] and computer games [4]. While many VEs rely on a visual and audio feedback, the use of haptic interface is known to greatly improve immersivity and to increase situational awareness of the operator [5]. Since haptic devices constitute controlled dynamical systems interacting with both VE and the operator, their use often arises questions regarding stability and transparency of the resulting haptic interface. These questions become especially challenging in the case of networked virtual environments interacting with multiple operators. Such systems are usually referred to as shared virtual environments (SVEs) [6] or collaborative virtual environments [7] and are in the focus of this paper.

![Schematic diagram of SVE with two haptic interfaces](image)

Fig. 1. Shared virtual environment setup with two haptic interface devices

Schematic diagram of SVE with two haptic interfaces is depicted in Fig. 1. In this setup the controller of each interface device has two-directional communication with VE and a two-fold goal. On the one hand, it needs to process and transfer information from the device to VE. This is in order to allow the influence of the operator on the environment. On the other hand, it needs to control the interface device using the information received from VE. This is in order to reflect the environment’s state to the operator via the “force feedback”. Note that while being attached to an SVE the dynamics of the two interface devices become coupled through the environment. Note also that in many practical applications, delays in communication are inevitable and can not be neglected. The need to coordinate haptic devices in the presence of communication delays falls into a category of challenging decentralized control problems and is a subject of the current paper.

A commonly used passivity-based approach to the control of multiple haptic devices for SVE relies on the notion of scattering transformation [8], [9]. This approach was developed in the context of delayed bilateral teleoperation systems [8] and has been extended for problems with multiple haptic devices [10] including those related to SVEs [11]. A remarkable advantage of the scattering transformation is that it guarantees delay-independent stability of the system. Note, however, that it restricts the controller structure and does not provide us with a convenient framework for optimizing system performance. This shortage seems to be immanent also in other passivity-based methods [12] as well as in the studies based on experimental or qualitative observations [13], [14], [15].

In this work we aim at developing an optimization-based framework for the decentralized controller synthesis using the notion of quadratic invariance [16]. This notion was recently exploited in the solution of an optimal control problem associated with delayed bilateral teleoperation [17]. In [18] some of the results from [17] were extended for the case of cooperative teleoperation and in this paper we demonstrate how the ideas from [17] can be applied in the context of SVE. In particular, we show that even though, generally, the control problem associated with SVE may result in a not tractable optimization, allowing indirect communication between the interface devices would result in a quadratically invariant control setup. This allows us to come up with an insightful structure possessed by all stabilizing controllers and to reduce the $H^2$ optimization to a number of uncoupled problems with multiplicative delay that can be handled using recent loop shifting techniques [19].

The remainder of this paper is organized as follows. Section II is devoted to the description of the control architecture, the global $H^2$ optimization to a...
problem associated with SVE. In Section III, we introduce the control architecture guaranteeing quadratic invariance of the problem. In Section IV parameterization of all stabilizing controllers is derived and it is shown that any stabilizing controller can be implemented within a structure that guarantees delay independent stability of the system. In Section V, we outline the solution of the $H^2$ optimization and present the optimal controller structure.

a) Notation: Linear fractional transformation is denoted by

$$F_l\left( \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, B \right) := A_{11} + A_{12}B(I - A_{22}B)^{-1}A_{21}.$$  

Positive and negative feedback operators are denoted by $T_p(A, B) := A(I - BA)^{-1}$ and $T_n(A, B) := A(I + BA)^{-1}$. Note that $C = T_p(A, B) \Leftrightarrow A = T_n(C, B)$. Delay operator of the length $h$ is denoted by $D(h) = e^{-sh}$.

II. SHARED VIRTUAL ENVIRONMENT WITH HAPTIC INTERFACES AS A CONTROL PROBLEM

Consider the schematic representation of a shared virtual environment setup depicted in Fig. 1. Note that the virtual environment there is assumed to be divided into two parts. The modelled part incorporates all components with linear time-invariant dynamics that may be included into a control problem formulation. The unmodelled part represents the rest of the virtual world and will be considered as a disturbance applied to the modelled part. To illustrate this concept let us consider the following example, which will be developed then along the whole paper.

Example: Consider a setup in which two operators are interacting with an object that continuously changes its size either randomly or with respect to some complex time varying processes simulated within the virtual environment. Assume that the motion is restricted to 1 degree of freedom and that the interaction with the object in the virtual world is implemented through some simulated dynamical systems that we will refer to as “end-effectors”. This situation is schematically depicted in Fig. 2. The virtual end-effectors are assumed to have known mass, stiffness and damping coefficients denoted with $m_v$, $k_v$ and $c_v$ respectively. The forces applied to them will be denoted by $f_1$, $f_2$ and their displacements will be denoted with $x_{v1}$, $v_{i2}$.

For illustrative purposes, we will model each of the interface devices as a 2nd order system presented in Fig. 3. The coefficients $m_r$, $k_r$ and $c_r$ account for both the device and the operator arm dynamics. The operator and control forces applied to the $i$'th interface device and its displacement will be denoted by $F_i$, $u_i$ and $x_i$ respectively.

A natural architecture for the control of two haptic interfaces for a shared virtual environment is depicted in Fig. 4. In this block diagram, the block $P : \begin{bmatrix} z_{p1} \\ z_{p2} \end{bmatrix} = \begin{bmatrix} P_{d1} & P_{f11} & P_{f12} \\ P_{d2} & P_{f21} & P_{f22} \end{bmatrix} \begin{bmatrix} d \\ f_1 \\ f_2 \end{bmatrix}$ represents the virtual environment, where $d$ is the disturbance from its unmodelled part, $f_1$, $f_2$ are signals received from the external ports and $z_{p1}$, $z_{p2}$ contains signals to be coupled with the first and the second interface devices respectively (displacements, forces, etc.) The blocks $G_i : \begin{bmatrix} y_i \\ z_i \end{bmatrix} = \begin{bmatrix} G_{iyF} & G_{iyu} \\ G_{iyF} & G_{iyu} \end{bmatrix} \begin{bmatrix} F_i \\ u_i \end{bmatrix} \quad \text{for } i = 1, 2$ represent interface device models, where $F_i$ are the forces applied by human operators, $u_i$ are the control signals, $y_i$ are the available measurements and $z_i$ contains the signals to be coupled with the virtual environment. Coupling at this point can be reflected by the magnitude of the signals $z_{p1} - z_1$ and $z_{p2} - z_2$, and the control goal can be defined as keeping these signals small. Controllers for the interface devices are represented for $i = 1, 2$ by $C_i : \begin{bmatrix} v_{ii} \\ f_{ii} \end{bmatrix} = \begin{bmatrix} C_{iyy} & C_{iyd} \\ C_{iyF} & C_{iyd} \end{bmatrix} \begin{bmatrix} y_i \\ d \end{bmatrix}$, where $d := Dd$ represents the delayed version of the disturbance, i.e., the signal received from the virtual environment site. Communication delays between the interface devices and the virtual environment are denoted by $D = e^{-sh}$, where $h$ is the delay length.

Remark 1: In this work, equal communication delays are assumed for the sake of simplicity. Note, however, that the results of this paper can be extended for the case of different delay lengths in the different communication channels.
Example: Let us continue with the example associated with Fig 2. In this setup \( P \) models the behaviour of the virtual environment subject to forces \( f_i \) and the object size \( d \). The outputs of \( P \) may be defined as \( z_{pi} = [ f_i ] \) for \( i = 1, 2 \). The \( G_i \) blocks model the "joystick" behaviour subject to the operator and the control forces. Their outputs are defined as \( z_i = [ f_i ] \) for \( i = 1, 2 \). In this setup the performance criteria \( \begin{bmatrix} z_{p1} - z_1 \\ z_{p2} - z_2 \end{bmatrix} \) corresponds to the positioning and force coupling between each "joystick" and the corresponding end-effector of the virtual environment.

At this point, we may cast the control scheme depicted in Fig. 4 as a generalized control setup presented in Fig. 5 with

\[
\begin{align*}
    w &= \begin{bmatrix} d \\ F_1 \\ F_2 \end{bmatrix}, \\
    u &= \begin{bmatrix} f_1 \\ f_2 \\ u_1 \\ u_2 \end{bmatrix}, \\
    y &= \begin{bmatrix} d \\ y_1 \\ y_2 \end{bmatrix}, \\
    z &= \begin{bmatrix} z_{p1} - z_1 \\ z_{p2} - z_2 \end{bmatrix}
\end{align*}
\]

and

\[
G = \begin{bmatrix}
    P_{d1} - G_{1z} & 0 & 0 \\
    0 & G_{2y} & 0 \\
    0 & 0 & G_{1y} \\
\end{bmatrix},
\]

\[
K = \begin{bmatrix}
    D^2 C_{1fd} & D C_{1fy} & 0 \\
    D^2 C_{2fd} & D C_{2fy} & 0 \\
    D C_{1ud} & D C_{1uy} & 0 \\
    D C_{2ud} & D^2 C_{2uy} & 0 \\
\end{bmatrix}.
\]

As one can be observe from (2), communication delays and limitations associated with the considered control architecture impose structural constraints on the design parameter \( K \). This is inline with the decentralized nature of the problem, which, in our view, is the main factor complicating control of multiple haptic interfaces to virtual environment over delayed communication.

III. QUADRATIC INVAARIANT CONTROL ARCHITECTURE

Decentralized control problems, such as the problem described in the previous section, substitute a serious theoretical challenge and no analytical solutions are currently available for the general case. In this work we consider a possibility to overcome this difficulty via conceptual modification of the control architecture as it shown in Fig. 6, where additional communication paths are indicated with blue. The conceptual difference of the modified setup from the original one is that the interface devices are allowed to share local measurements via their communication to the virtual environment. Note that in this situation it is natural to allow the measurements from each of the interface devices to influence both of the virtual environment ports, as manifested by the blocks \( K_{1fy} \) and \( K_{2fy} \).

This modified control architecture can be cast as a generalized control setup shown in Fig. 5 with

\[
G = \begin{bmatrix}
    P_{d1} - G_{1z} & 0 & 0 & 0 \\
    0 & G_{2y} & 0 & 0 \\
    0 & 0 & G_{1y} & 0 \\
    0 & 0 & 0 & G_{2y} \\
\end{bmatrix},
\]

\[
K = \begin{bmatrix}
    C_{1fd} & C_{1fy} & K_{1fy} \\
    C_{2fd} & C_{2fy} & K_{2fy} \\
    C_{1ud} & C_{1uy} & D^2 K_{1uy} \\
    C_{2ud} & D^2 K_{2uy} & C_{2uy} \\
\end{bmatrix}.
\]

Note that in this case the composite measured and control signals in Fig. 5 should be interpreted as

\[
u' = \begin{bmatrix} \hat{f}_1' \\ \hat{f}_2' \\ u_1' \\ u_2' \end{bmatrix},
\]

\[
y' = \begin{bmatrix} \hat{d} \\ y_1' \\ y_2' \end{bmatrix},
\]

with \( \hat{f}_i := e^{s_h f_i} \) for \( i = 1, 2 \) and \( \hat{d} := e^{-s_h d} \). Note also that in spite of the additional communication channels, communication delays still impose constraints on the controller structure and the setup remains within the class of decentralized problems. It turns out, however, that this modified decentralized problem possess a favourable property.

Let us denote the set of all the structurally admissible controllers, namely, of all \( K \) having a structure as in (4) by \( S \). It can be verified at this point that \( \forall K \in S : KG_{22} K \in S \), i.e., that the generalized control setup with \( G \) and \( K \) as in (3) and (4) is quadratically invariant, [16]. It will be shown below that this property greatly facilitates the controller design. In particular, it allows the use of Youla parameterization [20], which provides us with an important insight into the structure and properties possessed by all stabilizing controllers.

IV. DELAY INDEPENDENT STABILIZATION AND STABILIZING CONTROLLER STRUCTURE

Let us consider the modified control architecture depicted in Fig. 6 and the associated generalized control setup with \( G \) an \( K \) given in (3) and (4). Using the quadratic invariance property and assuming that the original dynamics of the master and slave devices are stable, the set of all internally stabilizing controllers can be characterized as

\[
K = T_n(Q, G_{22}), \quad \forall Q \in H^\infty \cap S_2.
\]
Note that (5) can be represented by the block diagram depicted in Fig. 7. Without deriving an explicit expression for $K$, we can use this block diagram to rewrite (5) as $u = Q(y - G_{22}u)$. Define $v := (y - G_{22}u)$ and denote its component by $v = \begin{bmatrix} \tilde{d} & v_1' & v_2' \end{bmatrix}$ with $v_i := y_i - G_{iyy}u_i$ for $i = 1,2$.

At this point, any stabilizing controller can be rewritten as $u = Qv$ or more explicitly as

$$\begin{bmatrix} \dot{f}_1 \\ \dot{f}_2 \\ \dot{u}_1 \\ \dot{u}_2 \end{bmatrix} = \begin{bmatrix} Q_{1fd} & Q_{1fy} & \Theta_{1fy} \\ Q_{2fd} & \Theta_{2fy} & Q_{2fy} \\ Q_{1ud} & Q_{1uy} & \Theta_{1uy}^2 \\ Q_{2ud} & D^2\Theta_{2uy} \end{bmatrix} \begin{bmatrix} \tilde{d} \\ v_1' \\ v_2' \end{bmatrix}, \quad (6)$$

where $Q_*$ and $\Theta_*$ are used to denote the sub-blocks of $Q \in \mathcal{S}$. This, in turn, allows to implement the controller as shown in Fig. 8, where

$$Q_* : \begin{bmatrix} u_{i} \\ f_{ii} \end{bmatrix} = \begin{bmatrix} Q_{iuy} & Q_{iuf} \\ Q_{iuf} & Q_{iud} \end{bmatrix} \begin{bmatrix} v_{i} \\ \tilde{d} \end{bmatrix}$$

for $i = 1,2$. To get in intuitive interpretation for this structure, note that $v_1$ and $v_2$ can be interpreted as the parts of the outputs $y_1$ and $y_2$ driven by the external signals $F_1$ and $F_2$, respectively. Thus, the distinctive feature of the stabilizing control architecture in Fig. 6 is that all the communication in it is based on signals containing information about external inputs only.

It follows immediately from the derivation that the structure in Fig. 6 remains valid for any constant communication delay $h$. This leads us to the main result of this section.

**Theorem 1:** Consider the control architecture depicted in Fig. 6 with some given stable LTIs $P_i, G_1, G_2$ and a constant communication delay $h > 0$. Then:

1) Any set of stabilizing controllers for this setup can be implemented as it shown in Fig. 8 for some $Q_*$ and $\Theta_*$ in $H^\infty$.

2) The control scheme depicted in Fig. 8 is stable for any $Q_*$ and $\Theta_*$ in $H^\infty$ and for an arbitrary large constant communication delay $h$.

The theorem above suggests that the controller structure presented in Fig. 8 can be considered as an alternative to the passivity-based control schemes. Note, however, that unlike the passivity based methods, in the proposed approach the controller structure follows naturally from the Youla parameterization and does not restrict the controller choice. In other words, we show that any stabilizing controller implemented in an appropriate way guarantees delay-independent stability.

**Example:** To demonstrate delay-independent stability of the proposed control architecture, let us present numerical simulations for the example described in Section II. As a first step, we choose system coefficients as $m_v = 4kg$, $c_v = 10Ns/m$, $k_v = 260Nm$, $m_r = 3kg$, $c_r = 10Ns/m$, $k_r = 200Nm/m$ and introduce dynamic weights for the force and position tracking errors and for the control actions $\begin{bmatrix} \frac{1}{200} & -0.5 \\ -0.5 & \frac{1}{200} \end{bmatrix}$ and $\begin{bmatrix} 0.004 \pm 0.004 \\ 0.004 \pm 0.004 \end{bmatrix}$, respectively. Then, using standard methods, we calculate an LQG controller for the delay-free case of the problem and implement it once within the natural architecture (Fig. 6) and once within the stabilizing architecture (Fig. 8). Exploring the influence of an increasing delay length on the behaviour of the resulting systems, one may realize that for the delay over 0.5 sec the system with natural controller implementation looses stability. On the other hand, controller implemented within the stabilizing architecture maintains stability for any delay length. To demonstrate this, a stable behaviour of the system with 0.5 sec communication delay is depicted in Fig. 9.

**V. OPTIMIZATION AND OPTIMAL CONTROLLER STRUCTURE**

Let us go back to the architecture depicted in Fig. 6 and the associated generalized setup presented in Fig. 5 with $G$ and $K$ given in (3) and (4) respectively. Our goal now is to find an internally stabilizing controller $K \in \mathcal{S}$ that minimizes the $H^2$ norm of $\mathcal{F}_1(G,K)$. Note that applying
parameterization of all stabilizing controllers, (5), the closed-loop transfer matrix can be conveniently expressed in terms of the Youla parameter as $\mathcal{F}(G, K) = G_{11} + G_{12} Q G_{21}$ for $Q \in H^\infty \cap \mathcal{S}$. This way, the original control problem reduces to an equivalent affine optimization associated with the stabilizing controller structure shown in Fig. 8.

**OQ**: Given the transfer matrices $G_{11}, G_{12}, G_{21} \in RH^\infty$ as in (3), find $Q \in H^\infty \cap \mathcal{S}$ minimizing the $H^2$ norm of

$$ T = G_{11} + G_{12} Q G_{21} \quad (7) $$

The problem above falls into the category of model matching optimization problems with structural constraints on the design parameter. In the past few years a number of problems within this class were considered in the literature [21], [22], [23], [24]. In particular, a problem with structural constraints due to communication delays were studied in [25]. Even though, generally, model matching optimization with structural constraints may constitute a substantial theoretical challenge, the special case of this problem arising in the current work admits a relatively simple solution. The reason for this is in the block-diagonal structure of $G_{21}$, which corresponds to the fact that the external inputs applied to the different sites in our decentralized control setup are independent. This makes our problem similar to the one recently considered in [17] in the context of bilateral teleoperation. In particular, this allows to solve OQ columnwise.

Let us introduce a partitioning $T = \begin{bmatrix} T_1 & T_2 & T_3 \end{bmatrix}$ compatible with that of $G_{21}$. Note that since $G_{21}$ is block-diagonal each column of $T$ depends on a single column of $Q$ only. Therefore, using the fact that $||T||_2^2 = ||T_1||_2^2 + ||T_2||_2^2 + ||T_3||_2^2$, we may split the optimization column-wise into three independent problems. The problem associated with the first column can be interpreted as a problem of controlling the overall system basing on measurements from the unmodelled part of the virtual environment. After performing some row/column permutations in $G_{12}$ and in the first column of $Q$, this problem can be formulated as follows.

**OQ1**: Given the transfer matrices $P_{d1}, P_{d2}, P_{f_{11}}, P_{f_{12}}, P_{f_{21}}, P_{f_{22}}, G_{1zu}, G_{2zu} \in RH^\infty$ and $D = e^{-sh}$ for some $h > 0$, find $Q_{1ud}, Q_{2ud}, Q_{1fd}, Q_{2fd} \in H^\infty$ minimizing

$$ \begin{bmatrix} P_{d1} \\ P_{d2} \end{bmatrix} + D \begin{bmatrix} -G_{1zu} & 0 \\ 0 & -G_{2zu} \end{bmatrix} \begin{bmatrix} P_{f_{11}} & P_{f_{12}} \\ P_{f_{21}} & P_{f_{22}} \end{bmatrix} \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & D \\ 0 & 0 & 0 & D \end{bmatrix} \begin{bmatrix} Q_{1ud} \\ Q_{2ud} \\ Q_{1fd} \\ Q_{2fd} \end{bmatrix}_2. $$

The problem associated with the second column can be interpreted as a problem of controlling the overall system basing on measurements from the first haptic device. After performing some row/column permutations in $G_{12}$ and in the second column of $Q$, it can be formulated as follows.

**OQ2**: Given the transfer matrices $G_{1zf}, P_{f_{11}}, P_{f_{12}}, P_{f_{21}}, P_{f_{22}}, G_{1zu}, G_{2zu}, G_{1yk} \in RH^\infty$ and $D = e^{-sh}$, for some $h > 0$, find $Q_{1uy}, Q_{1fy}, \Theta_{2fy}, \Theta_{2uy} \in H^\infty$ minimizing

$$ \begin{bmatrix} -G_{1zf} \\ 0 \end{bmatrix} + D \begin{bmatrix} -G_{1zu} & P_{f_{11}} & P_{f_{12}} \\ 0 & P_{f_{21}} & P_{f_{22}} \end{bmatrix} \begin{bmatrix} Q_{1uy} \\ Q_{1fy} \\ \Theta_{2fy} \\ \Theta_{2uy} \end{bmatrix}_{22}. $$

Finally, the problem associated with the third column can be interpreted as a problem of controlling the overall system basing on measurements of the second haptic device $y_2$.

**OQ3**: Given the transfer matrices $G_{2zf}, P_{f_{11}}, P_{f_{12}}, P_{f_{21}}, P_{f_{22}}, G_{1zu}, G_{2zu}, G_{2yk} \in RH^\infty$ and $D = e^{-sh}$ for some $h > 0$, find $Q_{2uy}, Q_{2fy}, \Theta_{1fy}, \Theta_{1uy} \in H^\infty$ minimizing

$$ \begin{bmatrix} 0 \\ -G_{2zf} \end{bmatrix} + D \begin{bmatrix} 0 & P_{f_{11}} & P_{f_{12}} \\ -G_{2zu} & P_{f_{21}} & P_{f_{22}} \end{bmatrix} \begin{bmatrix} Q_{2uy} \\ Q_{2fy} \\ \Theta_{1fy} \\ \Theta_{1uy} \end{bmatrix}_{22}. $$

The sequence can now be concluded by noticing that OQ1, OQ2 and OQ3 can be efficiently solved using the recent loop-shifting techniques and that complete solutions of these problems can be derived by direct application of the results from [19]. In particular, it can be shown that solutions of these problems admit the following structures

$$
\begin{align*}
Q_{1ud} &= L_d \begin{bmatrix} \hat{Q}_{1ud} \\ \hat{Q}_{1fd} \end{bmatrix}, & L_d &= \begin{bmatrix} I & 0 & 1_{1u1} & 1_{1u2} \\ 0 & I \end{bmatrix}, \\
Q_{2ud} &= L_d \begin{bmatrix} \hat{Q}_{2ud} \\ \hat{Q}_{2fd} \end{bmatrix}, \\
Q_{1fd} &= L_1 \begin{bmatrix} \hat{Q}_{1fd} \\ \hat{Q}_{1fy} \end{bmatrix}, & L_1 &= \begin{bmatrix} I_{1u3} & 1_{1u4} & 1_{1u5} \\ 0 & 1 & 1_{1f1} \\ 0 & 0 & I \\ 0 & 0 & I \end{bmatrix}, \\
Q_{2fd} &= L_2 \begin{bmatrix} \hat{Q}_{2fd} \\ \hat{Q}_{2fy} \end{bmatrix}, & L_2 &= \begin{bmatrix} I_{2u3} & 1_{2u4} & 1_{2u5} \\ 0 & I & 1_{2f1} \\ 0 & 0 & I \\ 0 & 0 & I \end{bmatrix},
\end{align*}
$$

where $\hat{Q}_i$ are some finite-dimensional transfer matrices and $\Pi_i$ are FIR systems. It can be shown also that impulse responses of all the FIR systems in the above definitions have a support on $[0, b]$, except for $\Pi_{1u5}$ and $\Pi_{2u5}$, having support on $[0, 2h]$. In order to portray the resulting controller structure, let us define

$$
\begin{align*}
\hat{Q}_{1uy} &= \begin{bmatrix} \hat{Q}_{1uy} \\ \hat{Q}_{1fy} \end{bmatrix}, & \hat{Q}_{2uy} &= \begin{bmatrix} \hat{Q}_{2uy} \\ \hat{Q}_{2fy} \end{bmatrix}, \\
\hat{Q}_{1fy} &= \begin{bmatrix} \hat{Q}_{1fy} \\ \hat{Q}_{2fy} \end{bmatrix}, & \hat{Q}_{1fy} &= \begin{bmatrix} \hat{Q}_{1fy} \\ \hat{Q}_{2fy} \end{bmatrix},
\end{align*}
$$

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Denote the rows of $L_1$ and $L_2$ by $L_{1i}$ and $L_{2i}$ for $i = 1..4$, respectively. Finally, define
\[
L_{1d1} := \begin{bmatrix} I & \Pi_{1u2} & \Pi_{1u1} \end{bmatrix}, \quad L_{1d2} := \begin{bmatrix} 0 & 0 & I \end{bmatrix},
\]
where the partitioning is compatible with that in $\tilde{Q}_{1d}$, and
\[
L_{2d1} := \begin{bmatrix} I & \Pi_{2u2} & \Pi_{2u1} \end{bmatrix}, \quad L_{2d2} := \begin{bmatrix} 0 & 0 & I \end{bmatrix},
\]
with the partitioning compatible to that in $\tilde{Q}_{2d}$. At this point the structure of the $H^2$ optimal controller can be depicted as shown in Fig. 10. It is worth mentioning that since the optimization associated with the considered control problem is infinite-dimensional due to the presence of communication delays, the resulting optimal controller is infinite-dimensional as well. Note, however, that in the optimal controller structure presented in Fig. 10 all the infinite dimensional components are incorporated into the $L_i$ blocks and constitute implementable FIR systems.

**References**


