Soft-constrained model predictive control based on off-line-computed feasible sets

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Abstract—This paper explores an approach to softening of constraints in a class of model predictive control (MPC) algorithms that employ off-line-computed feasible sets for simplified online operations. The proposed approach relies on the use of an exact penalty function in order to ensure that the solution to the problem coincides with the actual optimal solution if the original MPC problem is feasible and that the there are minimum possible constraint violations if the original problem is infeasible. The approach is considered for a class of linear systems with multiplicative and additive disturbances, and its performance is analyzed for specific cases of non-stochastic and stochastic disturbances. The implementation of the approach with a dynamic-policy-based algorithm is also discussed.

Index Terms—Soft-constrained MPC, Stochastic MPC, Multiplicative and additive disturbances

I. INTRODUCTION

Disturbances are usually handled in constrained model predictive control (MPC) by employing nominal, stochastic or robust techniques. While stochastic techniques are preferred when the information on the distribution of the disturbances is available, a robust solution becomes the choice when only their bounds are known. In any case, an MPC algorithm may not necessarily guarantee the satisfaction of constraints at all times without being conservative, particularly if such a guarantee requires the designer to consider unreasonably larger bounds on disturbances. In this paper, we focus on reducing the conservativeness of an MPC algorithm for systems that permit an occasional breach of constraints by appropriately softening the constraints.

The softening of constraints in MPC has been widely researched (see, e.g., [1]–[4]). The essential idea used is to consider penalization of constraint violations instead of imposing hard constraints, thus allowing the solution of the MPC problem even when the hard-constrained problem is infeasible. In recent years, constraint-softening has been considered in the context of MPC under stochastic disturbances ( [5]–[12]) where the robust approach is not particularly appealing because of possibly conservative results. In most papers on stochastic MPC, constraints are softened either by imposing the constraints on the average of the quantities being constrained (e.g., [9], [13]) or by considering a probabilistic satisfaction of constraints (e.g., [6], [11]). Such an approach is reasonable while dealing with stochastic disturbances; however, it does not readily allow appropriate penalization of individual constraint violations whenever such violations become necessary.

In this work, we explore constraint-softening in a nominally or probabilistically robust MPC scheme for a class of linear systems with both multiplicative and additive disturbances which may be stochastic. The scheme is based on the use of an off-line-computed feasible (constraint-admissible) set which is nominally or probabilistically contractive, and it allows the softening of constraints fulfilling twin objectives: one, of ensuring that the solution to the control problem is the same as the usual solution when the state is within the feasible set, and the other, of ensuring that the solution is obtained with minimum possible constraint violations when the state is outside the feasible set. The proposed scheme employs an appropriate exact penalty function (see, e.g., [14]) in the MPC optimization problem in order to meet the desired objectives.

The proposed approach guarantees that any system state outside the feasible set enters into it within a finite time under some nominal levels of disturbances. This basic idea is utilized to explore the probabilistic satisfaction of constraints when the disturbances are stochastic and their distributions are known. Although we analyze the handling of constraints in the case of stochastic disturbances by constructing a sequence of feasible sets in a way similar, in some sense, to the approaches used in [6], [7] etc., our control policy is a single-mode control policy defined by a continuous cost function over the relevant state-space, and it requires the solution of only a single, simple optimization problem at each time step. This can be useful in situations where the actual system initial state of the control problem is determined at a higher level based on factors including its own predicted cost. We discuss the implementation of the proposed approach with an MPC scheme that employs a class of dynamic control policies which incorporate all disturbance information into the input predictions [15]. Such policies are well suited for the class of uncertain systems we consider and result in simplified on-line computations. We assess the performance of the scheme with a numerical example.

Notations: $\mathbf{1}$ denotes a vector of all ones and $I$ denotes an identity matrix. For a scalar $\alpha$, $[\alpha]$ denotes the smallest integer not less than $\alpha$. For a vector $\mathbf{x}$, $\mathbf{x}(i)$ denotes its $i$th component and $\text{diag}(\mathbf{x})$ represents the diagonal matrix with the components of $\mathbf{x}$ as its diagonal elements. For vectors $\mathbf{x}$ and $\mathbf{y}$, $(\mathbf{x};\mathbf{y})$ represents $[\mathbf{x}^T \mathbf{y}]^T$. $\mathbb{Z}_+$ represents the set of non-negative integers. The inequality signs $\geq, \leq$ etc. denote positive/negative (semi-) definiteness of matrices. $\oplus$ denotes set addition and $\text{Co}$ denotes convex hull.
II. HARD-CONSTRAINED ROBUST MPC

We consider the linear time-varying system

\[ x_{k+1} = A_k x_k + B_k u_k + D_k w_k \]

with \([A_k B_k D_k] = [\bar{A} \bar{B} \bar{D}] + \sum_{i=1}^1 [A^{\Delta_i} B^{\Delta_i} D^{\Delta_i}] v_k[i] \]

where the state \(x_k \in \mathbb{R}^{n_x}\) and the input \(u_k \in \mathbb{R}^{n_u}\) are supposed to satisfy the constraint \((x_k; u_k) \in \Xi\), where \(\Xi\) is a C-set\(^1\), and the disturbances \(v_k \in \mathbb{R}^{n_v}\) and \(w_k \in \mathbb{R}^{n_w}\) are assumed to be bounded in C-sets \(\mathbb{V}\) and \(\mathbb{W}\) respectively.

A closed-loop MPC strategy for (1) involves the computation, at each time \(k\), of the decision variables represented by, say, a vector \(\xi_k\), from which predicted inputs \(u_{k+i|k}, i \in \mathbb{Z}_+\) are obtained using some rules \(u_{k+i|k} = K_0(x_k; \xi_k)\) and \(u_{k+i+1|k} = K_i(x_k; \xi_k, u_k, w_k)\), \(i > 0\), where \(\xi_k = (v_k[k]; u_{k+1|k}; \ldots; v_{k+i-1|k})\) and \(w_k = (w_k[k]; u_{k+1|k}; \ldots; w_{k+i-1|k})\). The on-line computation is aimed at minimizing a predicted cost function, usually of the form

\[
\sum_{i=0}^{\infty} \left( \|Q^{\frac{1}{2}} x_{k+i|k} \|^2 + \|R^{\frac{1}{2}} u_{k+i|k} \|^2 \right), \quad Q > 0, R > 0,
\]

(2)
evaluated for a nominal, worst-case or mean-square performance, and expressed as a convex function \(J(x_k, \xi_k)\) of \(x_k\) and \(\xi_k\). Strategies in which the rules \(K \equiv \{K_i(\cdot\cdot\cdot)\}_{i=0}^\infty\) are determined off-line usually allow the off-line computation of a constraint-admissible set \(\mathcal{Z} \subseteq \mathbb{R}_+^{n_x+n_u}\) for the vector \((x_k; \xi_k)\) so that \(\xi_k\) can be efficiently on-line by minimizing \(J(x_k, \xi_k)\) such that \((x_k; \xi_k) \in \mathcal{Z}\). Since \(v_k\) and \(w_k\) are possibly uncertain and unmeasurable, \(\mathcal{Z}\) must be robustly admissible in the sense that for any \((x_k; \xi_k) \in \mathcal{Z}, (x_{k+i|k}; u_{k+i|k}) \in \Xi\) holds for the predicted system for all \(i \in \mathbb{Z}_+\).

Let us consider a strategy specified by \(\xi_k, K, J(\cdot\cdot\cdot)\), and the corresponding robustly admissible set \(\mathcal{Z} \subseteq \mathbb{R}_+^{n_x+n_u}\). Also, let \(\mathcal{X} = \text{Proj}(\mathcal{Z}) = \{x \mid \exists \xi \text{ s.t. } (x; \xi) \in \mathcal{Z}\}\). Then, for any \(x_k \in \mathcal{X}\), the MPC optimization problem can be stated as

\[
\min_{\xi_k} J(x_k, \xi_k) \quad \text{subject to} \quad (x_k; \xi_k) \in \mathcal{Z}.
\]

(3)

We denote by \(\xi^*_k\) the optimal solution of (3) and define the optimal input at time \(k\) as \(u^*_k = \sigma(x_k) = K_0(x_k, \xi_k^*)\).

The MPC formulation of (3) is suitable with control parameterizations which allow an off-line computation of \(\mathcal{Z}\), and it has been considered for several practical problems (see, e.g., [6], [15–18]). The performance of an MPC scheme based on (3) will depend on the choice of \(J(\cdot\cdot\cdot)\). Typically, \(J(\cdot\cdot\cdot)\) is chosen to represent the nominal, worst-case or mean-square cost, thus leading to one of several possible performance choices such as the convergence of the state to a minimal set [19], input-to-state or \(\ell_\infty\) stability [15], \(\ell_2\) stability and mean-square stability [6].

III. SOFT-CONSTRAINED MPC USING AN EXACT PENALTY FUNCTION

We wish to devise a soft-constrained version of the MPC problem (3) to handle situations where \(v_k\) and \(w_k\) usually remain well inside the sets \(\mathbb{V}\) and \(\mathbb{W}\), and hence, practically, a fully robust solution to the problem (using a robustly admissible set \(\mathcal{Z}\)) may be too conservative.

A. Soft-constrained MPC formulation

Considering a strategy specified by \(\xi_k, K, J(\cdot\cdot\cdot)\), we construct an associated set \(\mathcal{Z}\) in the \(x\)-\(\xi\) space, satisfying

\[
(x_k; K_0(x_k, \xi_k)) \in \Xi \subseteq \Xi, \forall (x_k; \xi_k) \in \mathcal{Z} 
\]

and also ensuring \(x_{k+1} \in \mathcal{X} = \text{Proj}(\mathcal{Z})\) for any \((x_k; \xi_k) \in \mathcal{Z}\), either in some nominal sense (i.e., ensuring \(x_{k+1} \in \mathcal{X}\), \(\forall v_k \in \mathcal{V}, w_k \in \mathcal{W}\)) or in some probabilistic sense (i.e., ensuring \(\text{Pr}\{x_{k+1} \in \mathcal{X} \mid (x_k; \xi_k) \in \mathcal{Z}\} \geq p\) with a desired probability \(p\) when \(v_k\) and \(w_k\) are stochastic). We consider \(\mathcal{Z}\) as the baseline set which is safe with regards to constraint satisfaction, and envisage \(\Xi\) as a design option which allows different safety margins in \(\mathcal{Z}\) w.r.t. different constraints (e.g., a higher margin for input constraints ensuring no violations in an expanded set \(c\Xi\), for some \(c > 1\)).

We make the following assumptions about \(\mathcal{Z}\) and \(J(\cdot\cdot\cdot)\).

A1. \(\mathcal{Z}\) is a polyhedral C-set in \(\mathbb{R}_+^{n_x+n_u}\) and is defined as \(\mathcal{Z} = \{(x; \xi) \mid M_x x + M_\xi \xi \leq 1\}\), where \(M_x \in \mathbb{R}^{m_x \times n_x}\) and \(M_\xi \in \mathbb{R}^{m_\xi \times n_\xi}\).

A2. \(J(x; \xi)\) is convex and quadratic, defined as \(J(x; \xi) = x^T P_1 x + 2x^T P_2 \xi + \xi^T P_3 \xi\) for \(x, \xi \in \mathcal{Z}\) where 

\[
\Xi = \{x \mid \exists \xi \text{ s.t. } (x; \xi) \in \mathcal{Z}\},
\]

(4)

With \(\mathcal{Z}\) constructed as discussed above, problem (3) may not be feasible for all \(k \in \mathbb{Z}_+\) even if \(x_0 \in \mathcal{X}\). So, we look for a soft-constrained version of (3) satisfying the requirements:

R1. If \(x_k \notin \mathcal{X}\), the modified problem gives the same solution as problem (3).

R2. If \(x_k \notin \mathcal{X}\), the solution is such that \((x_k, \xi^*_k) \in \sigma \mathcal{Z}\) where \(\sigma = \inf \{\sigma \mid \exists \xi \text{ s.t. } (x_k; \xi) \in \sigma \mathcal{Z}\}\).

R1 is standard and has been considered in earlier works such as [2], [3] where it is achieved by using exact penalty functions. \(R2\) is motivated by the desire to limit constraint violations to the minimum possible level at each time step.

For the stated requirements, we consider the modified version of problem (3) in the following form:

\[
\min_{\xi_k, \sigma} J(x_k, \xi_k) + \alpha h(\sigma) \quad \text{subject to} \quad (x_k; \xi_k) \in \sigma \mathcal{Z},
\]

(5)

where \(\alpha\) is a positive scalar and \(h(\cdot)\) is a suitably chosen function. Now, we note the following result regarding an exact penalty function (see, e.g., [14, Sec. 5.4.5]).

Lemma 1: Given an optimization problem

\[
\min_{\xi_k} f(\xi) \quad \text{subject to} \quad g_i(\xi) \leq 0, i = 1, \ldots, m
\]

with convex and differentiable functions \(f(\xi)\) and \(g_i(\xi)\), \(i = 1, \ldots, m\), the optimal solution set of the following problem

\[
\min_{\xi_k} f(\xi) + \alpha \epsilon \quad \text{subject to} \quad g_i(\xi) \leq \epsilon, i = 1, \ldots, m
\]

will be the same as that of (6) if strict duality holds for (6) and \(\alpha\) in (7) satisfies \(\alpha > 1/\lambda^*\), where \(\lambda^*\) is an optimal solution of the dual problem \(\max_{\lambda \geq 0} q(\lambda)\) with

\[
q(\lambda) = \min_{\xi_k} f(\xi) + \sum_{i=1}^m \lambda_i g_i(\xi).
\]
Lemma 1 essentially presents a way to define an appropriate exact penalty function for a constrained optimization problem such that minimizing the penalty function solves the original constrained problem.

Now, we let \( \lambda_2^*(x_k) \) denote the optimal solution to the Lagrange dual of problem (3) for a given \( x_k \in \mathcal{X} \), and define
\[
\Lambda = \max_{x \in \mathcal{X}} \mathbf{1}^T \lambda_2^*(x).
\] (8)

Proposition 2: Requirements R1-R2 will be fulfilled for any \( x_k \in c\mathcal{X} \), \( c \geq 1 \) by (5) with \( h(\sigma) = \sigma \) if \( \alpha > c \Lambda \).

Proof: Given the convexity and differentiability of \( J(\ldots) \), strong duality holds for problem (3) for any \( x_k \in \mathcal{X} \) since \( \mathcal{Z} \) is polyhedral and hence constraints are all affine in \( \xi_k \) and the problem is feasible. Since the constraint \( (x_k; \xi_k) \in \sigma \mathcal{Z} \) in (3) is represented by \( M_x x_k + M_{\xi \xi_k} \leq \sigma \mathbf{1} \), and since problem (5) with \( h(\sigma) = \sigma \) will have the same solution as with \( h(\sigma) = \epsilon - 1 \) for a constant \( \alpha \), the fulfillment of R1 with \( h(\sigma) = \sigma \) and \( \alpha > c \Lambda \geq \Lambda \) directly follows from Lemma 1. Next, to see that R2 is fulfilled for any \( x_k \in c\mathcal{X} \), let \( \sigma_k \leq c \) be the constant as defined in R2 for the given \( x_k \). Then, let us consider the problem
\[
\min_{\xi_k} J(x_k; \xi_k) \quad \text{subject to} \quad (x_k; \xi_k) \in \sigma_k \mathcal{Z},
\] (9)
which is feasible since \( x_k \in \sigma_k \mathcal{X} \). We now show that, for the given \( x_k \), any optimal solution of (5) with \( h(\sigma) = \sigma \) is also an optimal solution of (9) if \( \alpha > c \sigma_k \Lambda \).

Let us consider problem (3) for a system state \( \bar{x}_k = x_k / \sigma \in \mathcal{X} \), and let \((\xi_\bar{x}^*, \lambda_\bar{x}^*)\) be an optimal primal-dual solution pair to this problem. Since strong duality holds, the solution pair satisfies the Karush-Kuhn-Tucker (KKT) conditions which can be written as:
\[
\begin{align*}
\lambda_\bar{x}^* &\geq 0 \\
M_\xi \xi_\bar{x}^* + M_{\xi \xi} \bar{x}_k &\leq \mathbf{1} \\
\text{diag}(\lambda^*) (M_\xi \xi_\bar{x}^* + M_{\xi \xi} \bar{x}_k - \mathbf{1}) &\geq 0 \\
2P_{22} \xi_\bar{x}^* + 2P_{12} \bar{x}_k + M_{\xi \lambda} \lambda^* &\geq 0
\end{align*}
\] (10)

Multiplying both sides of the equations in (10) by \( \sigma_k \) and denoting \( \xi_\bar{x}^* \sigma_k \) and \( \lambda_\bar{x}^* \sigma_k \) respectively by \( \xi_k^* \) and \( \lambda_k^* \), we get
\[
\begin{align*}
\lambda_\bar{x}^* \sigma_k &\geq 0 \\
M_\xi \xi_k^* + M_{\xi \xi} x_k &\leq \mathbf{1} \\
\text{diag}(\lambda^*) (M_\xi \xi_k^* + M_{\xi \xi} x_k - \mathbf{1}) &\geq 0 \\
2P_{22} \xi_k^* + 2P_{12} x_k + M_{\xi \lambda} \lambda_k^* &\geq 0
\end{align*}
\] (11)

which imply that \( \xi_k^* \) and \( \lambda_k^* \) form an optimal primal-dual solution pair for problem (9). Then, since \( \sigma_k \) is a constant for the given \( x_k \), so that problem (5) with \( h(\sigma) = \sigma \) will have the same solution as with \( h(\sigma) = \sigma - \sigma_k \), it follows from Lemma 1 that a solution of (5) with \( h(\sigma) = \sigma \) is also a solution of (9) if \( \alpha \) in (5) satisfies \( \alpha > 1 / \lambda \lambda_\bar{x}^* \sigma_k \) where \( 1^T \lambda^* = \lambda_\bar{x}^*(\bar{x}_k) \) and \( \lambda^*_\bar{x}^*(\bar{x}_k) \) is upper bounded by \( \Lambda \) for any \( \bar{x}_k \in \mathcal{X} \). Hence, for any \( x_k \in \mathcal{X} \), \( c > 1 \), the fulfillment of R2 by the solution of (5) with \( h(\sigma) = \sigma \) follows if \( \alpha > c \Lambda \geq \sigma_k \Lambda \).

To use the result of Proposition 2, we need to determine \( \Lambda \) over \( \mathcal{X} \). The following result shows that it is sufficient to look for the largest dual value over the boundary of \( \mathcal{X} \).

Lemma 3: Let \( \partial \mathcal{X} \) denote the boundary of the set \( \mathcal{X} \). Then,
\[
\Lambda = \max_{x \in \partial \mathcal{X}} \mathbf{1}^T \lambda_2^*(x).
\] (12)

Proof: This lemma can be proved by showing that for any \( x \in \partial \mathcal{X}, \lambda_2^*(\alpha x) < \lambda_2^*(x) \) for any positive \( \rho < 1 \). Details are omitted.

Note that the solution to (3) for a polyhedral \( \mathcal{Z} \) is piecewise affine over polyhedral partitions of \( \mathcal{X} \) [4], and hence over partitions of \( \partial \mathcal{X} \). Since the same set of constraints are active over a partition, observing from (10), it is obvious that over a particular partition, the optimal dual value will have the largest 1-norm at one of its vertices. Hence, \( \Lambda \) can be found by checking the dual values at these vertices. Further, the maximum value of \( c \) to be used to find \( \alpha \) in (5) can be decided by considering the absolute bounds of \( w_k \) and \( v_k \).

B. Performance of soft-constrained MPC

In this section, we assess the performance of the MPC algorithm based on the online solution of problem (5), especially with regard to the satisfaction of constraints when the state is outside or close to the boundary of the set \( \mathcal{X} \). Essentially, the performance for system states outside \( \mathcal{X} \) mirrors that for the states on its boundary. This allows us to control and analyze this performance in terms of the (possible) movement of the state towards the set. In the sequel, we assume that \( K_0(x_k, \xi_k) \) is linear in \( \xi_k \) and \( x_k \).

1) System with non-stochastic disturbances: In the case non-stochastic disturbances, it is not possible to give any reasonable results on the levels of constraint violations if the disturbances remain unpredictable. So, given an \( x_k \in \mathcal{X} \), we discuss the eventual satisfaction of constraints when the disturbances return into the nominal sets \( \mathcal{V} \) and \( \mathcal{W} \).

Lemma 4: For any \( x_k \notin \mathcal{X} \), if \( w_{k+i} \in \mathcal{W} \) and \( u_{k+i} \in \mathcal{W} \), \( i = 0, 1, \ldots, \) an MPC algorithm based on (5) ensures that the state enters \( \mathcal{X} \) within \( N \) steps, where \( c_1 \) and \( c_0 \) are the smallest numbers satisfying \( \text{Rea}_{1, 0}(x_k) \subseteq c_1 \mathcal{X} \) and \( \text{Rea}_{1, \infty}(x_k) \subseteq c_0 \mathcal{X} \), \( c_2 = c_1 - c_0 \), and \( \text{Rea}(c) \) denotes the reachable set.

Proof: Let \( K_0(x_k, \xi_k) = L_x x_k + L_{\xi \xi} \xi_k \). Then, for any \( x_k \notin \mathcal{X} \), we have
\[
\begin{align*}
x_{k+1} &= (A_k + B_k L_{x_k}) x_k + B_k L_{\xi \xi} \xi_k^* + D_k w_k \\
&= (A_k + B_k L_{x_k}) x_k + \sigma_k B_k L_{\xi \xi} \xi_k^* + D_k w_k \\
&\quad + (\sigma_k - 1) \{(A_k + B_k L_{x_k}) x_k / \sigma_k + B_k L_{\xi \xi} \xi_k^* / \sigma_k \} \\
&\in c_1 \mathcal{X} \cup \{(x_k - 1) \sigma_k \mathcal{X} = \sigma_k (c_0 + c_2) \mathcal{X} = x_k, \mathcal{X} \}
\end{align*}
\] (13)

Similarly, the next state \( x_{k+2} \) will satisfy
\[
x_{k+2} \in (c_{01} + c_{02}) \mathcal{X} = \sigma_k \mathcal{X} = (c_{01} + c_{02}) \mathcal{X}
\]

Proceeding in the same way, we have \( x_{k+N} \in \mathcal{X} \), \( N \) where \( \mathcal{X} = \sigma_k \mathcal{X} = \sigma_k (c_0 + c_2 + c_4) \mathcal{X} \).
Hence, the state \( x_{k+N} \) will be inside \( X \) if \( \sigma x_0 + c \Delta^N 1-c^N \leq 1 \) and hence the lemma follows. \( \blacksquare \)

The bound on \( N \) in Lemma 4 is only a conservative upper bound which can be adjusted by ensuring a suitable value of \( c_1 \) (and \( c_0 \)) during the synthesis of the control policy.

2) System with stochastic disturbances: Let us assume that \( v_k, k = 0, 1, \ldots \) and \( w_k, k = 0, 1, \ldots \) are independent and identically distributed (i.i.d.) random vectors with means \( \mathbb{E}(v_k) = 0 \) and \( \mathbb{E}(w_k) = 0 \), and covariance matrices \( \mathbb{E}(v_k v_k^T) = I \) and \( \mathbb{E}(w_k w_k^T) = \sum_n^3 \). Also, let us assume that the disturbances are finitely improbable, i.e., \( v_k \in \mathbb{V} \) and \( w_k \in \mathbb{W}, k \in \mathbb{Z}_+ \). We now wish to express and impose constraints in terms of the long-term probability of their complete satisfaction and the probabilities of their violations at different levels.

We start by defining a probabilistically contractive set \( \mathcal{Z} \). Given a probability \( p \), let \( \mathcal{Z} \) be such that, for \( X = \operatorname{Proj}_\mathcal{Z} \), the control law \( \kappa(.) \) associated with problem (5) ensures that

\[
\Pr\{x_{k+1} \in cX | x_k \in X\} \geq p
\]

(14)

for some positive \( c < 1 \). Condition (14) implies that there exist, for each \( x \in X \), disturbance-sets \( \mathcal{V}(x) \) and \( \mathcal{W}(x) \) satisfying \( \Pr\{(v_k; w_k) \in \mathcal{V}(x) \times \mathcal{W}(x)\} \geq p \) holds. Also, let \( c_1 \) be the smallest \( c \) satisfying (15) with the same \( \mathcal{V}(x) \) but with \( \mathcal{W}(x) = \{0\} \), \( \forall x \in X \) and define \( c_\Delta = c_1 - c_0 \) as in Section III-B. Also, assume that the relation \( \operatorname{Rea}_{(\alpha, \nu, \mathcal{W})}(c\mathcal{X}) \subseteq cX \) is satisfied for some \( c \geq 1 \) and \( \bar{c} \) be the smallest such \( c \).

Now we divide the state space into several partitions, and model the inclusion of the state in these partitions and its transition from one partition to another as a discrete Markov chain. In this context, we define a sequence of sets as

\[
\mathcal{X}_i = \rho_i \mathcal{X}, \rho_i = \frac{1 - c \Delta^i 1-c_\Delta}{c_0}, i = 0, 1, \ldots, N_\mathcal{E}
\]

(16)

where \( N_\mathcal{E} \) is the smallest \( i \) such that \( \rho_i \geq \bar{c} \). Further, we define \( N = N_\mathcal{E} + 1 \) partitions of \( \mathcal{X} \), as

\[
\mathcal{X}_i^\Delta = \begin{cases} \mathcal{X}_{i-1}^\Delta \backslash \mathcal{X}_{i-2}^\Delta & \text{for } i = 1 \\ \mathcal{X}_{i-1}^\Delta & \text{for } i = 2, 3, \ldots, N \end{cases}
\]

(17)

The partitions defined in (17) provide a way to assess the probabilistic transition of the state into the set \( \mathcal{X} \) from outside it. We represent the inclusion of the state in the set \( \mathcal{X} \) different partitions by \( N \) states of a Markov chain.

Lemma 5: For each partition \( \mathcal{X}_i^\Delta, i = 1, \ldots, N \), let \( m_i \geq i \) be the smallest integer such that \( \operatorname{Rea}_{(\alpha, \nu, \mathcal{W})}(\mathcal{X}_i^\Delta) \cap \mathcal{X}_i^\Delta \neq \emptyset \), and let a matrix \( T \) of transition probabilities be specified with elements \( t_{ij} \) defined for \( i = 1, \ldots, N \) as

\[
t_{ij} = \begin{cases} p & \text{if } j = i \text{ or } j = i - 1 \\ (1-p)f_{ij} & \text{if } j = i, \ldots, m_i \\ 0 & \text{otherwise} \end{cases}
\]

(18)

We do not consider non-stationary disturbances here, though they may be handled in some cases, at least possibly with some added conservativeness.

where, for each \( i \in \{1, 2, \ldots, N\} \), \( f_i \in \mathbb{R}^{m_i-1} \) is a vector of the form \([0 \ 0 \ 0 \ 1]^T \). Then, (a) we have the relation \( \lim_{k \to \infty} \frac{1}{n} \sum_{k=0}^n \Pr\{x_k, u_k \in \mathbb{Z}\} \geq \pi h, \) where \( \pi \) is the unique row vector satisfying \( \pi = \pi T \), \( \pi^T \mathbf{1} = 1 \), and \( h \) is a vector whose first element is 1 and the rest 0; and (b) given that \( x_k \in \mathcal{X}_i^\Delta \), the relation \( \frac{1}{n} \sum_{k=0}^n \Pr\{x_{i+1}, u_{i+1} \in \mathbb{Z}\} \geq p_{cs} \), holds for any \( p_{cs} < \pi h \) and any \( L > L = \inf \{k | e_i^T \mathbf{1} \geq p_{cs}, \forall r \geq k\} \), where \( e_i \) is the vector with 1 as its \( i \)th element and 0 as the rest of the elements.

Proof: First note that the structure of \( T \) in (18) is such that the Markov chain is irreducible and aperiodic. So, a unique steady-state distribution \( \pi \) satisfying \( \pi = \pi T \) exists [20]. Next, from (13), note that \( \Pr\{x_{i+1} \in (\sigma x_0 + c \Delta^N) \geq p \) holds for any \( x_k \) outside \( X \) and hence we have \( \Pr\{x_{i+1} \in \mathcal{X}_{i-1} \mid x_k \in \mathcal{X}_i\} \geq p \). However, since we approximate the transition probabilities in \( T \) by assuming \( \Pr\{x_{i+1} \in \mathcal{X}_i \mid x_k \in \mathcal{X}_i\} = p \) and \( \Pr\{x_{i+1} \in \mathcal{X}_i \mid x_k \in \mathcal{X}_i\} = 1-p \), the quantity \( \pi h \) only bounds the steady state probability of the state lying in \( \mathcal{X}_i^\Delta \). Hence, part (a) follows. Further, for any \( p_{cs} < \pi h \), \( L \) is finite and the relation in part (b) follows from the definition of \( L \) since \( L \geq L \).

The above result can be readily extended to estimate probabilities of various levels of constraint violations.

Remark 1: The bound \( \pi h \) in Lemma 5 can be improved to some extent by choosing more realistic transition probabilities in \( T \); for instance, choosing more appropriate vector \( \bar{f}_i, \bar{f}_i \mathbf{1} = 1 \), in (18) better distributing the outward transition probabilities. However, the bound will remain conservative since the probabilities may vary widely over each partition and the worst-case condition may rarely be applicable for more than a few times for a state-trajectory. One way to better approximate the transitions to and from \( \mathcal{X}_i^\Delta \) is to consider further partitions within \( \mathcal{X} \). This can be done by allowing \( \sigma \) in (5) to be less than 1 down to some desired level \( \sigma_{min} \) such that the set \( \sigma_{min} \mathcal{Z} \) is still probabilistically contractive by some small factor \( c_{min} \). As will be shown with an example in Section V, it will shift the steady-state probability profile more towards the inside of \( \mathcal{X} \).

Remark 2: Obtaining \( \mathcal{Z} \) and \( \mathcal{X} \) that satisfy (14) without conservativeness for a given \( p \) is usually not so straightforward. One way is to first obtain them assuming \( \mathcal{V}(x) = \mathcal{W} \) and \( \mathcal{W}(x) = \mathcal{W} \) which satisfy \( \Pr\{(v_k; w_k) \in \mathcal{V} \times \mathcal{W}\} \geq \bar{p} \) with some \( \bar{p} \), and then to estimate \( p \) through realistic simulations. Since the estimated \( p \) will likely be larger than \( \bar{p} \), one can start with a relatively low value of \( \bar{p} \).

IV. IMPLEMENTATION OF SOFT-CONSTRAINED MPC USING OPTIMIZED DYNAMIC POLICY

We consider an input parameterization of the form

\[
u_{k+i+1} = K x_{k+i+1} + \delta_{k+i+1}, \quad i \in \mathbb{Z}_+ \]

(19a)

where \( K \) is a precomputed feedback gain optimal in some sense, and \( \delta_{k+i+1}, i \in \mathbb{Z}_+ \) represent predicted perturbations to the input \( K x_{k+i+1} \). Here, \( \delta_{k+i+1}, i \in \mathbb{Z}_+ \) are parameterized as the predicted future outputs of a controller system

\[
\xi_{k+i+1} = G_k \xi_k + F_k w_k, \quad d_k = H_k \xi_k
\]

(19b)

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where $\xi_k \in \mathbb{R}^{n_k}$ is the controller state. The controller matrices $G_k$ and $F_k$ are assumed to follow the same kind of time-variation as the system matrices, i.e., at any time $k \in \mathbb{Z}_+$,

$$[G_k F_k] = [\bar{G} \bar{F}] + \sum_{i=1}^{\eta} [G^{\Delta_i} F^{\Delta_i}] v_k[i],$$

(19c)

where $\bar{G}$, $\bar{F}$, $G^{\Delta_i}$, $F^{\Delta_i}$, $i = 1, \ldots, \eta$ are design matrices. The input policy of (19) follows the parameterization proposed in [15], and the incorporation of both the multiplicative and additive disturbances in the perturbation dynamics of (19b) is motivated by the effectiveness of uncertainty/disturbance-based input parameterizations (e.g., [8], [17], [21]) which lead to less conservative but tractable MPC problems.

Determining the sequence $\{w_k[i/k] \}_{i \in \mathbb{Z}_+}$ in real time, based on the parameterization in (19), ideally, requires an optimization of the matrices $H$, $G$, $F$, $G^{\Delta_i}$, $F^{\Delta_i}$, $i = 1, \ldots, \eta$ and the controller state $\xi_k$ at each time $k$. However, adopting the procedure proposed in [15], we simplify the online computations by computing the controller matrices and the set $Z$ off-line so that only $\xi_k$ remains to be optimized at each time $k$. Assuming that the set $\bar{\Psi}$, in which the disturbance $v_k$ lies nominally or probabilistically, is a polytopic set defined by vertices $\bar{v}^{(r)}$, $j = 1, \ldots, \nu$, we can see that the augmented system

$$\zeta_{k+1} = \Psi_k \zeta_k + D_k w_k$$

(20)

with $\zeta_k = (x_k; \xi_k)$, $\Psi_k = [A_k + B_k K_k B_H^H \bar{G}_k]$, and $D_k = [D_k]$ is a polytopic system, i.e., $[\Psi_k \ D_k] \in \text{Co} \{[\bar{\Psi}^{(r)} \ D^{(r)}] : j = 1, \ldots, \nu \}$ where $\bar{\Psi}^{(r)} = \left[ \begin{array}{c} \bar{\Psi}_0 + B_k H_k \bar{G}_k \mid \sum_{i=1}^{\eta} \left[ A_i^{\Delta_i} B_i^{\Delta_i} \right] \bar{v}_i^{(r)} \end{array} \right]$ and $D^{(r)} = \left[ D_0 + \sum_{i=1}^{\eta} D_i^{\Delta_i} \bar{v}_i^{(r)} \right]$. So, assuming that the set $\bar{\Psi}$ is a polytopic set defined by the vertices $\bar{v}^{(r)}$, $j = 1, \ldots, \tau$, we can apply the methods of [15] to determine the controller matrices by maximizing the projection on the $x$-subspace of an ellipsoidial set $Z$ for $\zeta_k$ such that $\text{Re} \bar{\Psi}_{[v; 0]}(Z) \subseteq \bar{e}_1 Z$ for some positive $\bar{e}_1 < 1$ and that $(x; K x + H x) \in \bar{Z}$, $\forall(x; \xi) \in Z$. Once the controller matrices are computed, the polyhedral constraint-admissible set corresponding to $Z$ can be used as the set $Z$ in (5). In any case, $Z$ should satisfy (4) and, with a chosen $\bar{e}_1$, the contraction condition

$$\text{Re} \bar{\Psi}_{[v; 0]}(Z) \subseteq \bar{e}_1 Z, \quad \text{Re} \bar{\Psi}_{[0; v]}(Z) \subseteq \bar{e}_1 Z, \quad \bar{e}_0 \leq \bar{e}_1$$

(21a)

in the case of non-stochastic disturbances or

$$\text{Re} \bar{\Psi}_{[v; 0]}(Z) \subseteq \bar{e}_1 Z, \quad \text{Re} \bar{\Psi}_{[0; v]}(Z) \subseteq \bar{e}_0 Z, \quad \bar{e}_0 \leq \bar{e}_1$$

(21b)

for some $\Psi(\zeta)$ and $W(\zeta)$ such that $\text{Pr} \{ (v_k; w_k) \in \Psi(\zeta) \times W(\zeta) \} \geq p$, $\forall \zeta \in Z$ in the case of stochastic disturbances.

With $Z$ obtained as mentioned above, the MPC algorithm employing an on-line solution of (5) with a $J(\ldots)$ that approximates the cost in (2) ensures that the controlled system has a response as discussed in Lemma 4 or Lemma 5 with some $c_1 \leq \bar{c}_1$ and $c_0 \leq \bar{c}_0$ when the system state is outside or close to the boundary of the set $X$. However, since we allow the control predictions to depend on disturbances, it may be desirable to guarantee a relevant steady-state performance as well. In the case of non-stochastic disturbances, we may ensure a nominal or an $H_{\infty}$-like performance as in [15]. With

$$\begin{align*}
\text{stochastic disturbances, one may look for a condition of the form}
\lim_{L \to \infty} \frac{1}{L} \sum_{k=0}^{L} & \mathbb{E} \left( \| Q^\frac{1}{2} x_k \|^2 + \| R^\frac{1}{2} u_k^* \|^2 \right) \\
& \leq \bar{\theta}
\end{align*}

(22)

for some $\bar{\theta}$. Such performances can be achieved if the controller matrices and the cost matrix satisfy certain conditions such as those that ensure the $\ell_2$ or mean-square stability of the predicted closed-loop system (see, e.g., [7], [15]. However, since the cost in (5) includes an additional term $\alpha h(\sigma)$, in order to ensure that the system under MPC achieves the desired performance, we may need to impose, at each $k$, an additional constraint that $J(x_k; \xi_k) \leq \bar{\theta}$ is not greater than any possible value of the predicted cost $J(x_k; \xi_{k-1})$.}

V. NUMERICAL EXAMPLE

For a brief illustration, we consider a system described by

$$A = \left[ \begin{array}{ccc} 1 & 1.05 & 1.05 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{array} \right], \quad B = \left[ \begin{array}{c} 1 \\ 0 \\ 0 \end{array} \right], \quad D = \left[ \begin{array}{c} 1 \\ 0 \\ 0 \end{array} \right]$$

$$A^{\Delta_1} = \left[ \begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & 0.05 \\ 0 & 0 & 0.05 \end{array} \right], \quad B^{\Delta_1} = \left[ \begin{array}{c} 0.05 \\ 0.05 \\ 0.05 \end{array} \right], \quad D^{\Delta_1} = 0$$

with $\bar{\Psi} = \{ v \mid |v| \leq 1 \}$ and $\bar{W} = \{ w \mid \| w \|_{\infty} \leq 1 \}$, and the constraints $|u_k| \leq 1$, $|x_k[1]| \leq 30$, and $|x_k[2]| \leq 5$. The controller matrices are obtained using the LQ optimal $K$ and imposing $\bar{c}_1 = 0.98$, and the cost matrix is obtained for a nominal performance. $\Lambda$ is computed to be equal to 22000 and we consider (5) with $\alpha = 2 \Lambda$. Simulations are carried out for $x_0 = (-54.4, 7.5)$ which is outside $X$, $v_k$ is chosen uniformly randomly from $\bar{\Psi}$ whereas $w_k = \text{sign}(x_k)$ is chosen. The resulting state trajectory is shown in Fig. 1a (solid line). The solid line in Fig. 1b shows the value, at each $k$, of the smallest $\sigma$ such that $x_k \in \sigma X$. These values coincide with the optimal values of $\sigma$ obtained from (5). Fig. 1 also shows the plots for two alternative settings: the first using $\alpha = \Lambda$ instead of $\alpha = 2 \Lambda$, and the second using $\sigma = \sigma_{x_0} = 1.8$ in (5). The plots show that constraint violations at higher levels are more limited with $\alpha = 2 \Lambda$ than with $\alpha = \Lambda$. When the constraint violation is not penalized, the maximum allowed violations continue for a significant time.

Next, for the same system, we assume $v_k$ to be a zero-mean Gaussian disturbance of unit variance and each component of $w_k$ to follow the uniform distribution between $-2$
and 2. We choose the set $\bar{V}$ such that $\Pr(v_k \in \bar{V}) = 0.8$ and choose $\bar{W}$ equal to $0.5\bar{W}$ so that $\Pr(w_k \in \bar{W}) = 0.25$. The controller matrices are computed considering LQ optimal $\mathbf{K}$ and imposing $\bar{\epsilon}_1 = 0.98$. It is found that $N_\epsilon = 30$ is sufficient to ensure the invariance of the set $\rho_{\text{in}}\mathcal{X}$ in the presence of disturbances $v_k \in \bar{V}$ (truncated at 98% confidence level) and $w_k \in \bar{W}$. The cost matrix $\mathbf{P}$ is determined such that $J(x_k, \xi_k)$ is a stochastic Lyapunov function and represents the expected cost for the predicted closed-loop system with $w_{k+d} = 0$, $i \in \mathbb{Z}_+$. Note that no controller matrices could be found with a parameterization as in [6] or even with (19) without the feedforward of $w_k$. (i.e., with $F_k = 0$).

Parameter $p$ was re-estimated through simulations and was found to be about 0.89 — significantly larger than the value used in the synthesis. Matrix $\mathcal{T}$ was obtained by choosing $f_{k+j}^i$ to be proportional to the cross-sectional widths of the reachable outer partitions. In Fig. 2a, the plot marked with small stars shows the steady-state probability of inclusion of the state within the set $\sigma \mathcal{X}$. It can be observed that while the probability of the state remaining within $\mathcal{X}$ (and hence the input remaining within $U$) is 0.72, the probability of the state remaining within 110% of the normal bounds is 0.98. This steady-state probability profile is compared with the cases where we consider $2$ and $5$ further nested sets inside $\mathcal{X}$ (plots marked with circles and squares). It is obvious that the profile moves more inside the set $\mathcal{X}$ increasing the probability of the state remaining within $\mathcal{X}$ to 0.83 and 0.93 respectively.

We also assessed the control performance of the proposed scheme considering 1000 different realizations of disturbances for an initial state $x_0 = (30.5, -5.1)$ which lies inside $\mathcal{X}$ close to its boundary. State trajectories were observed for 15 time steps for the three cases mentioned above with identical sets of disturbance realizations. Fig. 2b shows the number of constraint violations at various time steps in the three cases. Although violations were found to occur up to the 9th time step for some realizations; the average violations were less than 2 times per trajectory in all the cases.

VI. Conclusion

In this paper, we have analyzed constraint-softening for a class of MPC algorithms that use off-line-computed polyhedral feasible sets. The soft-constrained MPC algorithm retains the simplicity of the original hard-constrained MPC problem as well as the continuity of the cost function while satisfying two desired requirements: that of giving the same solution when the original problem is feasible and of ensuring minimal constraint violations when the hard-constrained problem is not feasible. We have explored the performance of the approach for specific cases of non-stochastic and stochastic disturbances and illustrated it with a brief example.

REFERENCES


