Lie algebras and regularity of controls for real-analytic control systems

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Abstract—We prove, for real-analytic control-affine control systems, that whenever a control \( \eta \) and corresponding trajectory \( \xi \) are such that the terminal point of \( \xi \) belongs to the boundary of the attainable set from the initial point of \( \xi \), it follows that the control \( \eta \) is real-analytic on an open dense subset of its interval of definition. Furthermore, for every trajectory-control pair \((\xi, \eta)\) such that \( \xi \) starts at a point \( x_0 \) and ends at a point \( x_1 \), it is possible to find a (possibly different) trajectory-control pair \((\xi', \eta')\) such that \( \xi' \) also goes from \( x_0 \) to \( x_1 \) and the control \( \eta' \) is real-analytic on an open dense subset of its interval of definition. Similar results are proved for time-optimal controls. Our theorems improve upon results proved before for the time-optimal control case, and the proofs illuminate much more clearly the role of the Lie algebras of vector fields associated to these problems.

I. INTRODUCTION

The purpose of this paper is to prove some general results on regularity of trajectories for real-analytic single-input systems that are affine in the control, improving upon earlier work by the author, cf. [3]. Furthermore, the proof is much simpler, and displays clearly the roles of both Lie algebraic conditions and real analyticity.

The result of [3] was for time-optimal trajectories. Here we present a general result for boundary trajectories, showing that for any such trajectory the corresponding control must be real analytic on an open dense subset of its interval of definition. This is, acknowledged, a very weak regularity property, but it is decidedly nontrivial, and it is remarkable that it is completely universal, in the sense that it is valid for all systems in our class, subject only to the condition that the data vector fields be real-analytic.

II. DIFFERENTIATING THE SWITCHING FUNCTION

The method to be used is the old and well known approach of “successively differentiating the switching function until the control appears explicitly, and then solving for the control.” That is, for a trajectory \( \xi \) of a system \( \dot{x} = f(x) + u g(x) \), if \( \lambda \) is the Pontryagin adjoint vector, then the switching function \( \sigma \) is given by

\[
\sigma(t) = \langle \lambda(t), g(\xi(t)) \rangle,
\]

and it is well known that \( \sigma \) “determines the control \( \eta' \)”, in the sense that \( \eta(t) \) must be \(+1\) whenever \( \sigma(t) > 0 \), and \(-1\) whenever \( \sigma(t) < 0 \). So, if \( \sigma \) only has, for example, finitely many zeros, then the control is completely determined and smooth on the complement of a finite set.

One then often studies the more general situation where the interval \([0,T]\) of definition of our control and trajectory is divided into intervals \( I_j \) such that on each \( I_j \) the function \( \sigma \) is either never zero or identically zero. Then on the intervals where \( \sigma \) is nonzero the control is smooth, and on the intervals where \( \sigma \equiv 0 \) one differentiates the equation \( \sigma(t) = 0 \), obtaining an equality of the form \( \sigma_1(t) + \eta(t) \sigma_2(t) = 0 \). If “the control appears explicitly”, that is, if \( \sigma_2(t) \neq 0 \), then one can solve for the control, obtaining

\[
\eta(t) = -\frac{\sigma_1(t)}{\sigma_2(t)},
\]

from which one hopes to be able to conclude that the control is smooth.

If “the control does not appear explicitly”, that is, if \( \sigma_2(t) \equiv 0 \), then one can differentiate this condition, and repeat the procedure.

This method, as described, has several well know drawbacks. For example:

(1) There is no reason, in general, why a partition into intervals \( I_j \) as above should exist. The set of points \( t \) where \( \sigma(t) \neq 0 \) is indeed open, so it is a finite or countable union of open intervals. But the set of points \( t \) where \( \sigma(t) = 0 \) is, in principle, just an arbitrary closed set, and could be rather wild, for example, a nowhere dense set of positive measure.

(2) There is a need for an “organizing principle” that would enable us to describe the successive functions such as \( \sigma_2 \) that arise from the differentiations in a systematic way.

(3) It is not clear that the process of successive differentiations might not continue forever, always yielding new derivatives where the coefficient of the control vanishes identically.

In this paper we go some way towards taking care of these difficulties, and manage to prove some positive results, and what makes this possible is the systematic use of real analyticity and Lie algebraic conditions.

Lie algebraic conditions provide the “organizing principle” alluded to in Item (2) above. It turns out, by writing the adjoint equation of the Pontryagin Maximum Principle in an
intrinsic differential-geometric way, that the coefficients that occur in the successive differentiations of \( \sigma \) are, precisely, Lie brackets. For example, the derivative of \( \sigma(t) \)—i.e., of \( \langle \lambda(t), g(\xi(t)) \rangle \)—is \( \langle \lambda(t), [f, g](\xi(t)) \rangle \), which must thus vanish on any interval where \( \sigma \) vanishes. Then differentiation of the condition

\[
\langle \lambda(t), [f, g](\xi(t)) \rangle \equiv 0
\]

yields the condition

\[
\langle \lambda(t), [f, [f, g]](\xi(t)) \rangle + \eta(t)\langle \lambda(t), [g, f, g](\xi(t)) \rangle \equiv 0.
\]

On any interval where one of the functions

\[
t \mapsto \langle \lambda(t), [f, g]](\xi(t)) \rangle
\]

or

\[
t \mapsto \langle \lambda(t), [g, f, g]](\xi(t)) \rangle
\]

is nowhere zero, one can “solve for the control” and prove that the control is smooth. On any interval where one of these functions vanishes identically, one can differentiate further, and obtain a new expression involving the control and high-order brackets of \( f \) and \( g \). It turns out that, in this way, all iterated brackets of \( f \) and \( g \) eventually show up.

Could this procedure continue forever? The answer turns out to be “no” in the real-analytic case, and this is shown by using the existence property of maximal integral submanifolds, together with the nontriviality condition of the Pontryagin Maximum Principle. Precisely, the nontriviality condition tells us that the adjoint covector \( \lambda(t) \) must be nonzero when paired against some tangent vector at \( \xi(t) \), and what we need to know is that \( \lambda(t) \) is nonzero when paired against some \textit{iterated bracket} of \( f \) and \( g \) evaluated at \( \xi(t) \). This will follow if the linear span of all the iterated brackets, evaluated at a point, is the full tangent space of the state space at that point. In the real-analytic case, we can always assume that this is the case, because we can restrict ourselves to a maximal integral submanifold of the Lie algebra of vector fields generated by \( f \) and \( g \). (In the \( C^\infty \) case, it may happen that such an integral submanifold does not exist, and this explains why our results are not true in that case.)

We now move on to developing the details of the approach outlined above.

III. LIE ALGEBRAIC BACKGROUND

If \( M \) is a real-analytic manifold, we use \( VF^\omega(M) \) to denote the set of all real-analytic vector fields on \( M \). If \( \mathcal{V} \) is a subset of \( VF^\omega(M) \), then \( L(\mathcal{V}) \) will denote the Lie algebra of vector fields generated by \( \mathcal{V} \), that is, the smallest real linear space of vector fields on \( M \) that contains \( \mathcal{V} \) and is closed under the Lie bracket operation. We then let \( L_0(\mathcal{V}) \) denote the ideal of the Lie algebra \( L(\mathcal{V}) \) generated by all the differences \( X - Y \), \( X, Y \in \mathcal{V} \). It is well known that the manifold \( M \) is partitioned into \textit{maximal integral manifolds} of \( L(\mathcal{V}) \), that is, connected immersed (but not necessarily embedded) real analytic submanifolds \( S \) of \( M \) such that

(i) the tangent space \( T_xS \) of \( S \) at each point \( x \) of \( S \) is the space \( L(\mathcal{V})(x) \), where

\[
L(\mathcal{V})(x) \equiv \{ V(x) : V \in L(\mathcal{V}) \},
\]

and

(ii) any connected immersed real analytic submanifold \( S' \) of \( M \) that contains \( S \) as an open submanifold and for which \( T_xS' = L(\mathcal{V})(x) \) for all \( x \in S' \) is necessarily equal to \( S \).

Furthermore, if \( S \) is any maximal integral submanifold of \( L(\mathcal{V}) \), then one of the following two possibilities occurs:

1. \( L_0(\mathcal{V})(x) = L(\mathcal{V})(x) \) for every \( x \in S \), in which case \( S \) is also a maximal integral manifold of \( L_0(\mathcal{V}) \).

2. \( L_0(\mathcal{V})(x) \) is a subspace of codimension one of \( L(\mathcal{V})(x) \) for every \( x \in S \), in which case \( S \) is partitioned into maximal integral submanifolds of \( L_0(\mathcal{V}) \), each one of which has dimension equal to \( \dim S - 1 \).

The reason for this dichotomy is that the codimension of the subspace \( L_0(\mathcal{V})(x) \) in \( L(\mathcal{V})(x) \) is either 0 or 1 for every \( x \), and in addition the functions \( S \ni x \mapsto \dim L(\mathcal{V})(x) \) and \( S \ni x \mapsto \dim L_0(\mathcal{V})(x) \) are constant on any maximal integral manifold \( S \) of \( L(\mathcal{V}) \), as explained, for example, in [1] and [2].

A maximal integral manifold \( S \) of \( L(\mathcal{V}) \) such that

\[
L_0(\mathcal{V})(x) = L(\mathcal{V})(x) \text{ for every } x \in S
\]

will be called \textit{nondegenerate}, and one for which (2) holds will be said to be \textit{degenerate}.

\textbf{Lemma 3.1:} Let \( \mathcal{V} \) be a set of real-analytic vector fields on \( M \), and let \( S \) be a degenerate maximal integral manifold of \( S \). Then there exists a unique real-analytic 1-form \( \omega \) on \( S \) such that \( \omega(X) = 1 \) for every \( X \in \mathcal{V} \) and \( \omega(X) = 0 \) for every \( X \in L_0(\mathcal{V}) \). Furthermore, the form \( \omega \) is closed.

\textbf{Proof.} Fix a member \( X_0 \) of \( \mathcal{V} \). Then at each point \( x \) of \( S \) the tangent space \( T_xS \) is the direct sum of the linear subspace \( L_0(\mathcal{V})(x) \) and the one-dimensional subspace \( \mathbb{R}X_0(x) \). So we may define a linear functional \( \omega_x : T_xS \to \mathbb{R} \) by letting \( \omega_x(X_0(x)) = 1 \) and \( \omega_x(v) = 0 \) for \( v \in L_0(\mathcal{V})(x) \). This linear functional then satisfies \( \omega_x(X) = 1 \) for every \( X \in \mathcal{V} \), because if \( X \in \mathcal{V} \) then \( X - X_0 \in L_0(\mathcal{V}) \). The map \( \mathcal{V} \ni x \mapsto \omega_x \) is then the desired 1-form. Uniqueness is trivial.

To prove that \( \omega \) is closed, we use the well known formula

\[
d\omega(X, Y) = X\omega(Y) - Y\omega(X) - \omega([X, Y]),
\]

valid for arbitrary smooth vector fields \( X, Y \) on \( S \). Given \( x \in S \), and \( v, v' \in T_xS \), we can write \( v = X(x), v' = X'(x) \), where \( X = rX_0 + Y \) and \( X' = r'X_0 + Y' \), with \( r, r' \in \mathbb{R} \) and \( Y, Y' \in L_0(\mathcal{V}) \). Then \( \omega(X) \) and \( \omega(X') \) are the constant functions \( r, r' \), so \( X\omega(X') = X'\omega(X) = 0 \).

In addition, \( [X, Y'] = [rX_0 + Y, r'X_0 + Y'] \in L_0(\mathcal{V}) \), because \( [X_0, Y'], [X_0, Y'], \) and \( [Y, Y'] \) belong to \( L_0(\mathcal{V}) \). Hence \( \omega([X, Y']) = 0 \). So \( d\omega(X, X') = 0 \). In particular, \( d\omega(v, w) = 0 \) for arbitrary \( x \in S \) and \( v, w \in T_xS \), so \( d\omega \equiv 0 \), completing our proof.
IV. BOUNDARY TRAJECTORIES

We now study a control system $\Sigma$ of the form
\[ \dot{x} = f(x) + ug(x), \tag{1} \]
where $f$ and $g$ are real-analytic vector fields on a real analytic manifold $M$, and the controls take values in the interval $[-1, 1]$. A control for (1) is a measurable function with values in $[-1, 1]$, defined on an interval $[0, T]$. A trajectory of (1) for a control $\eta : [0, T] \mapsto [-1, 1]$ is an absolutely continuous curve $\xi : [0, T] \mapsto M$ such that $\xi(t) = f(\xi(t)) + \eta(t)g(\xi(t))$ for almost every $t \in [0, T]$. A trajectory-control pair of (1) is a pair $(\xi, \eta)$ such that $\eta$ is a control for (1) and $\xi$ is a trajectory of (1) for $\eta$.

To the system (1) we associate the two-vector-field set
\[ \mathcal{V} = \{ X, Y \} , \]
where $X = f - g$ and $Y = f + g$, and the Lie algebras $L(\mathcal{V})$ and $L_0(\mathcal{V})$. If $x \in M$, then the attainable set from $x$ is the set $A(x)$ consisting of all points $\xi(t)$ for all trajectory-control pairs $(\xi, \eta)$ of (1) such that $\xi(0) = x$, and all $t \in \text{domain}(\eta)$. It is well known that if $x \in M$ and $S$ is the maximal integral submanifold of $L(\mathcal{V})$ through $x$, then $A(x)$ is entirely contained in $S$. We define the boundary of $A(x)$ to be the set $\partial A(x)$ of all points of $A(x)$ that are not interior points of $A(x)$ relative to $S$, and we emphasize that with this definition $\partial A(x)$ is always a proper subset of $A(x)$, because the interior of $A(x)$ relative to $S$ is always nonempty, thanks to Chow’s Theorem. A trajectory-control pair $(\xi, \eta)$, defined on an interval $[0, T]$, will be called a boundary trajectory-control pair if $\xi(T)$ belongs to $\partial A(\xi(0))$.

If $\eta : [0, T] \mapsto [-1, 1]$ is a control, we use $U(\eta)$ to denote the set of all times $t \in [0, T]$ such that $\eta$ is real-analytic in a neighborhood of $t$. It is then clear that $U(\eta)$ is a relatively open subset of $[0, T]$.

Our main regularity result is then the following.

**Theorem 4.1:** If $(\xi, \eta)$ is a boundary trajectory-control pair, defined on an interval $[0, T]$, then $U(\eta)$ is a dense subset of $[0, T]$.

**Proof.** We may clearly assume, without loss of generality that $M$ itself is a maximal integral submanifold of $L(\mathcal{V})$. So at each point $x$ of $M$ the tangent space $T_xM$ is the linear span of the values of $x$ of all the iterated brackets of $f$’s and $g$’s, and this implies that $T_xM$ is the linear span of the values of $x$ of all the iterated brackets of the form
\[ f_1 = [f_{i_1}, [f_{i_2}, [f_{i_3}, \ldots, f_{i_n}]]], \]
for all finite sequences $I = (i_1, \ldots, i_n)$ of indices $i_j$ belonging to the set $\{1, 2\}$, where we let
\[ f_1 = f, \quad f_2 = g. \]

(It is well known that every iterated bracket $B$ of $f$ and $g$ belongs to the linear span of the $f_1$, as shown by the example of the formula
\[ [[f, g], [f, [f, g]]] = [f, [g, [f, [f, g]]]] - [g, [f, [f, g]]]. \]
The general case follows by repeated use of the Jacobi identity.)

Let $(\xi, \eta)$ be a boundary trajectory, and suppose that the domain of $\xi$ and $\eta$ is the interval $[0, T]$. Then the Pontryagin Maximum Principle implies that there exists a "nontrivial Hamiltonian Maximizing adjoint vector $\lambda$ along $(\xi, \eta)$ for which the Hamiltonian vanishes." Precisely, this means that

1. $[0, T] \ni t \mapsto \lambda(t)$ is an absolutely continuous map into the cotangent bundle $T^*M$ of $M$,
2. $\lambda(t)$ belongs to $T^*_M(w)$ (the cotangent space of $M$ at $\xi(t)$) for every $t \in [0, T]$,
3. $\lambda(t) \neq 0$ for every $t \in [0, T]$,
4. $\lambda(t), f(\xi(t)) + \eta(t)g(\xi(t)) = 0$,
5. $\eta(t) = 1$ if $\langle \lambda(t), g(\xi(t)) \rangle > 0$ and $\eta(t) = -1$ if $\langle \lambda(t), g(\xi(t)) \rangle < 0$ for almost every $t \in [0, T]$,
6. $\lambda$ satisfies the "adjoint equation".

The meaning of Condition (6) can be made precise in several (equivalent) ways, and we choose of all these ways the one that seems to us to be most natural for our problem, namely, the assertion that
\[ \frac{d}{dt}(\lambda(t), V(\xi(t))) = \langle \lambda(t), f, V(\xi(t)) \rangle + \eta(t)\langle \lambda(t), g, V(\xi(t)) \rangle \]
for every smooth vector field $V$ on $M$.

It follows from the nontriviality condition (3) that for every $t \in [0, T]$ there exist a positive integer $n$ and a multiindex $I = (i_1, i_2, \ldots, i_n)$ such that
\[ \langle \lambda(t), f_I(\xi(t)) \rangle \neq 0. \]

We let $\nu(t)$ be the smallest such $n$. Then the map $[0, T] \ni t \mapsto \nu(t)$ is a function on $[0, T]$ with values in the set of natural numbers. It is easy to see that $\nu$ is upper semicontinuous. If $t \in [0, T]$ then $\langle \lambda(t), f_I(\xi(t)) \rangle \neq 0$ for some multiindex $I$ of length $\nu(t)$, and then $\langle \lambda(s), f_I(\xi(s)) \rangle \neq 0$ for all $s$ in some open neighborhood $W$ of $t$, from which it follows that $\nu(s) \leq \nu(t)$ for all $s \in W$.

Now pick a point $t \in (0, T)$ and a positive real number $\varepsilon$. We will show that the interval $(t - \varepsilon, t + \varepsilon)$ contains a point of $U(\xi(t))$. Let $s_0$ be the minimum of the numbers $\nu(s)$, for $s \in (t - \varepsilon, t + \varepsilon)$, and let $s_0 \in (t - \varepsilon, t + \varepsilon)$ be such that $\nu(s_0) = s_0 = \nu(s)$. We will show that $s_0 \in U(\xi(t))$.

Since $\nu$ is upper semicontinuous, we may pick a positive real number $\delta$ such that $(s_0 - \delta, s_0 + \delta) \subseteq (t - \varepsilon, t + \varepsilon)$ and $\nu(s) \leq s_0$ for all $s \in (s_0 - \delta, s_0 + \delta)$. Then $\nu(s)$ is actually equal to $s_0$ for all $s \in (s_0 - \delta, s_0 + \delta)$. Fix a multiindex $I = (i_1, i_2, \ldots, i_n)$ of length $n_0$ such that $\langle \lambda(s_0, f_I(\xi(s_0))) \rangle \neq 0$. Then we may assume, after making $\delta$ smaller, if necessary, that
\[ \langle \lambda(s), f_I(\xi(s)) \rangle \neq 0 \]
for all $s \in (s_0 - \delta, s_0 + \delta)$.

Now assume that $n_0 = 1$, so $f_1$ is just $f$ or $g$. If $f_1$ is $g$, then $\langle \lambda(s), g(\xi(s)) \rangle \neq 0$ for all $s \in (s_0 - \delta, s_0 + \delta)$, so one of the inequalities
\[ \langle \lambda(s), g(\xi(s)) \rangle > 0, \]
\[ \langle \lambda(s), s, g(\xi(s)) \rangle < 0, \]
holds throughout the interval \((s_0 - \delta, s_0 + \delta)\). It follows that either \(\eta(s) \equiv 1\) or \(\eta(s) \equiv -1\) throughout \((s_0 - \delta, s_0 + \delta)\). Then \(\eta\) is real-analytic on \((s_0 - \delta, s_0 + \delta)\), implying that \(s_0 \in U(\eta)\).

Next, assume that \(n_0 > 1\). Then either \(f_1 = [f, f_1]\) or \(f_1 = [g, f_1]\), for some multivector \(J\) of length \(n_0 - 1\). Clearly, then,

\[
\langle \lambda(s), f_1(\xi(s)) \rangle = 0 \quad (3)
\]

for all \(s \in (s_0 - \delta, s_0 + \delta)\). Differentiation of (3) yields

\[
\langle \lambda(s), [f, f_1](\xi(s)) \rangle + \eta(s) \langle \lambda(s), [g, f_1](\xi(s)) \rangle = 0 . \quad (4)
\]

On the other hand,

\[
\langle \lambda(s), [g, f_1](\xi(s)) \rangle \neq 0 . \quad (5)
\]

(This is just a restatement of (2) if \(f_1 = [g, f_1]\), but it also follows from (2) if \(f_1 = [f, f_1]\), because in this case the fact that \(\langle \lambda(s), [f, f_1] \rangle \neq 0\), together with (4), imply (5).)

In view of (5), we can “solve (4) for \(\eta(s)\) on \((s_0 - \delta, s_0 + \delta)\), obtaining

\[
\eta(s) = - \frac{\langle \lambda(s), [f, f_1](\xi(s)) \rangle}{\langle \lambda(s), [g, f_1](\xi(s)) \rangle} , \quad (6)
\]

and this will enable us to prove that \(\eta\) is real-analytic on \((s_0 - \delta, s_0 + \delta)\).

The proof of this fact is as follows: consider the real-valued functions \(\varphi, \psi\) on the cotangent bundle \(T^*M\), given by

\[
\varphi(x, p) = \langle p, [f, f_1](\xi(s)) \rangle ,
\]

\[
\psi(x, p) = \langle p, [g, f_1](\xi(s)) \rangle .
\]

Then \(\varphi, \psi\) are real-analytic, and \(\psi\) is nonzero along the curve

\[
(s_0 - \delta, s_0 + \delta) \ni s \mapsto \Xi(s) \defeq (\xi(s), \lambda(s)) \in T^*M .
\]

So we may pick an open neighborhood \(S\) of \(\Xi\) in \(T^*M\) on which \(\varphi\) never vanishes, and observe that \(\Xi\) is a solution of the differential equation

\[
\hat{\Xi}(s) = F(\Xi(s)) - \frac{\varphi(\Xi(s))}{\psi(\Xi(s))} G(\Xi(s)) ,
\]

where \(F, G\) are the Hamilton vector fields that are the lifts of \(f, g\) to \(T^*M\). Since (7) is a differential equation with a real-analytic right-hand side, the solution \(\Xi\) is a real-analytic curve in \(T^*M\), and then the control \(\eta\) is real-analytic on the interval \((s_0 - \delta, s_0 + \delta)\), because (7) implies that

\[
\eta(s) = - \frac{\varphi(\Xi(s))}{\psi(\Xi(s))} . \quad (8)
\]

V. GENERAL TRAJECTORIES

Suppose \((\xi, \eta)\) is a general trajectory-control pair of our system. Then, obviously, \(\eta\) can be an arbitrary control, with no nontrivial regularity property. It turns out, however, that if our purpose is to steer a particular initial point \(x_0 = \xi(0)\) to a particular target point \(x_1 = \xi(T)\), then we can always modify the control so as to produce another trajectory-control pair \((\xi', \eta')\) that will still steer \(x_0\) to \(x_1\) (in, possibly, a different time) and be such that \(U(\eta')\) is an open dense subset of the interval of definition of \(\eta'\).

To prove this, we observe first that we have already established the desired conclusion in the case when \((\xi, \eta)\) is a boundary trajectory-control pair. (In that case, we have in fact shown that \(\eta\) itself must have the desired regularity property, so no modification of the control is necessary.)

It follows that we can assume that \((\xi, \eta)\) is an interior trajectory-control pair, in the sense that \(\xi(T)\) belongs to the interior (relative to the maximal integral submanifold \(S\) of \(L(V)\) that contains \(\xi\) of the attainable set \(A(x_0)\).

We will prove that in this case one can actually steer \(x_0\) to \(x_1\) by means of a control which is bang-bang with finitely many switchings.

First of all, it is clear that we may assume that \(S = M\), as we did in the previous section. Then we start from the target point \(x_1\) and consider the “backward system”

\[
\dot{x} = -f(x) - u(g)(x), \quad (9)
\]

whose trajectories are the same as those of our original system, but running backward in time. (Precisely, \([0, T] \ni t \mapsto \xi(t)\) is a trajectory of (9) if and only if \([0, T] \ni t \mapsto \xi(T - t)\) is a trajectory of our original system.)

We then apply the positive form of Chow’s theorem to this backward system, and conclude that for every strictly positive time \(\tau\) there exist a positive integer \(N\) and a sequence \(X = (X_1, X_2, \ldots, X_N)\) of members of the set \([-f - g, -f + g]\), such that the set

\[
A_X(\tau, x_1) = \{ e^{tX}x_1 : t \in \mathbb{R}_+, |t| \leq \tau \}
\]

has a nonempty interior, where

1. \(\mathbb{R}_+ = \{ r \in \mathbb{R} : r \geq 0 \}\),
2. for an \(N\)-tuple \(t = (t_1, \ldots, t_N) \in \mathbb{R}_+^N\), we write

\[
e^{tX} = e^{t_1X_1}e^{t_2X_2} \cdots e^{t_NX_N} ,
\]
3. if \(x \in M\) and \(X\) is a vector field on \(M\), then \(t \mapsto e^{tX}x\) is the integral curve of \(X\) that goes through \(x\) at time \(0\).

Let \(\Omega\) be an open subset of \(M\) such that \(x_1 \in \Omega\) and \(\Omega\) is a subset of \(A(x_0)\). (Recall that \(A(x_0)\) is the attainable set from \(x_0\) for the original system \(\Sigma\).) Choose a positive \(\tau\) and an \(N, X\) as above, but with \(\tau\) chosen so small that the set \(A_X(\tau, x_1)\) is a subset of \(\Omega\). Clearly, every point of \(A_X(\tau, x_1)\) can be steered to \(x_1\) by means of a control (for the original system \(\Sigma\)) which is bang-bang with at most \(N\) switchings. On the other hand, since \(A_X(\tau, x_1) \subseteq \Omega\), every point \(x\) of \(A_X(\tau, x_1)\) can be reached from \(x_0\) means of some trajectory-control pair \((\zeta, \theta)\). This is true, in particular, if we pick \(x \in W\), where \(W\) is a nonempty open subset of \(A_X(\tau, x_1)\). Pick such an \(x\) and \(W\), and then pick a trajectory-control pair \((\zeta, \theta)\) that steers \(x_0\) to \(x\). Then there is a sequence \(\{\theta_n\}_{n=1}^{\infty}\) of bang-bang controls (with finitely many switchings) that converges weakly to \(\theta\), so that the corresponding trajectories \(\zeta_n\) (starting at \(x_0\)) converge to \(\zeta\). Then, if \(T_n\) is the terminal time of \(\zeta_n\), the point \(\zeta_n(T_n)\) eventually belongs to \(W\). But
$W \subseteq A_X(\tau, x_1)$, so $\zeta_n(T_n)$ can be steered to $x_1$ by means of a bang-bang control. Combining the control $\theta_n$ that steers $x_0$ to $\zeta_n(T_n)$, and the one that steers $\zeta_n(T_n)$ to $x_1$, we get a bang-bang control that steers $x_0$ to $x_1$, as desired.

So we have proved:

Theorem 5.1: For the system $\Sigma$, whenever there exists a trajectory-control pair $(\xi, \eta)$ that goes from a point $x_0$ to a point $x_1$, it follows that there exists a trajectory-control pair $(\xi', \eta')$ that also goes from $x_0$ to $x_1$ and is such that the control $\eta'$ is real-analytic on an open dense subset of its interval of definition.

VI. TIME-OPTIMAL TRAJECTORIES

Similar arguments can be used to prove results about time-optimal trajectories. Recall that every trajectory of $\Sigma$ is entirely contained in a maximal integral manifold of the Lie algebra $L(V)$. We call a trajectory $\xi$ nondegenerate if the maximal integral submanifold containing $\xi$ is nondegenerate.

Theorem 6.1: Let $(\xi, \eta)$ be a time-optimal nondegenerate trajectory-control pair for the real analytic system $\Sigma$. Then the set $U(\eta)$ is dense in the interval of definition of $\eta$.

Proof. The argument is exactly the same as that of the proof of Theorem 4.1. We apply the Pontryagin Maximum Principle as in that proof, and obtain an adjoint vector $\lambda$ that satisfies all the conditions of that proof, except only that now we do not have the condition that the Hamiltonian function

$$t \mapsto \langle \lambda(t), f(\xi(t)) + \eta(t)g(\xi(t)) \rangle$$

vanishes identically. So in this case we work with the iterated brackets $f_1 = [f_1, [f_2, \ldots [f_n, \ldots]]]$ such that $f_n = g$. (These are the brackets that linearly span the Lie algebra $L_0(V)$.) The key point in our proof was the fact that for every $t$ there must exist a bracket $f_1$ such that $\langle \lambda(t), f_1(\xi(t)) \rangle \neq 0$. In the proof of Theorem 4.1, this was guaranteed by the facts that

1. We were considering all the $f_1$,
2. the linear span of all the $f_1$ is $L(V)$,
3. $L(V)(\xi(t))$ is the full tangent space $T_{\xi(t)}M$ (because of our assumption (made without loss of generality) that $M$ is an integral submanifold of $L(V)$),
4. The nontriviality condition guarantees that $\lambda(t)$ does not vanish on all the tangent vectors belonging to $T_{\xi(t)}M$.

In the case under consideration here, we are only considering the $f_1$ such that $f_n = g$. However, the nondegeneracy assumption guarantees that the linear span of these brackets, evaluated at $\xi(t)$, is the full tangent space $T_{\xi(t)}M$, and this suffices to imply that there must exist a bracket $f_1$ for which $\langle \lambda(t), f_1(\xi(t)) \rangle \neq 0$. Once this is established, the rest of the proof goes on exactly as in the proof of Theorem 4.1, and the proof of Theorem 6.1 is complete.

We now discuss the degenerate case, i.e., the case when the spaces $L_0(V)(x)$ are of codimension one in $L(V)(x)$.

In that case, we can still use the “replacement version” of our Theorem 5.1, and conclude that if $(\xi, \eta)$ is a time-optimal trajectory-control pair, then there exists a trajectory-control pair $(\xi', \eta')$ such that $\xi'$ has the same initial and terminal points as $\xi$, and $\eta'$ is real-analytic on an open dense subset of its domain. But we do not know yet that this new trajectory-control problem is also time-optimal. To establish this fact, we use Lemma 3.1. According to this result, there exists a closed 1-form $\omega$ such that

$$\langle \omega, f \rangle = 1,$$

and

$$\langle \omega, V \rangle = 0 \text{ for every } V \in L_0(V).$$

It follows from the fact that the form is closed that, locally, there exists a function $\varphi$ such that $d\varphi = \omega$, and then, if $\xi : [0, T] \mapsto M$ is a solution of $\xi = f(\xi(t)) + \eta(t)g(\xi(t))$ for some control $\eta$, we have $T = \int_0^T \omega = \varphi(\xi(T)) - \varphi(\xi(0))$, showing that if $\xi' : [0, T'] \mapsto M$ is any other trajectory with the same initial and terminal points as $\xi$ then necessarily $T' = T$. (The function $\varphi$ need not exist globally, but this does not matter, because the replacement of $\xi$ by $\xi'$ can be done locally, that is, we may divide $\xi$ into small arcs and do the replacement for each one.)

The final conclusion is the following result.

Theorem 6.2: Let $(\xi, \eta)$ be a time-optimal trajectory-control pair for the real analytic system $\dot{x} = f(x) + u g(x)$. Then there exists a trajectory-control pair $(\xi', \eta')$ such that $\xi'$ goes from the same initial point as $\xi$ to the same terminal point in the same time and is such that $\eta'$ is real-analytic on an open dense subset of its interval of definition.

REFERENCES