Networked $H_\infty$ Filtering of Semilinear Diffusion PDEs*
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Abstract—We design a network-based $H_\infty$ filter for semilinear diffusion partial differential equations over a rectangular domain under distributed in space measurements. The sampled in time measurements are sent to the sensor over a communication network. The objective is to enlarge the sampling time intervals, while preserving a satisfactory error system performance in the presence of variable network-induced delays. We suggest to divide the domain into a finite number of rectangular sub-domains, where sensing devices provide spatially averaged state measurements to be transmitted through communication network. Sufficient conditions in terms of Linear Matrix Inequalities (LMIs) for the internal exponential stability and $L_2$-gain analysis of the estimation error are derived via the time-delay approach to networked control systems and the direct Lyapunov-Krasovskii method. Numerical example illustrates the efficiency of the method.

I. INTRODUCTION

Many important plants, such as chemical reactors and heat transfer processes, are governed by partial differential equations and are often described by uncertain models. The existing results [1], [3] on robust control of uncertain distributed parameter systems extend the state space or the frequency domain $H_\infty$ approach and are confined to the linear case. It is thus of interest to develop consistent methods that are capable of providing the desired performance of distributed parameter systems in spite of significant model uncertainties. The LMI approach [2], [8] is definitely among such methods.

Networked Control Systems (NCSs), where the plant is controlled via communication network, became a hot topic. The introduction of communication network media brings great advantages, such as low cost, reduced weight, simple installation/maintenance and long distance control. Long distance estimation/control of chemical reactors or air polluted areas (that can be modeled by diffusion PDEs [11], [16]) is potentially of great interest. It is important to provide a stability and performance certificate in spite of the network imperfections: variable sampling intervals and communication delays. Three main approaches have been used to the NCSs: the discrete-time, the hybrid system and the time-delay approaches.

While there exists an extensive literature on network-based control of finite dimensional systems, there are only a few works on network-based control of PDEs. For linear parabolic systems, mobile collocated sensors and actuators were considered in [4]. The discrete-time approach to sampled-data control of linear time-invariant distributed parameter systems was developed in [13], [18]. A model-reduction-based approach to network-based control of semilinear distributed parameters systems was introduced in [17], where a finite-dimensional controller was designed on the basis of a finite-dimensional system that captures the dominant (slow) dynamics of the infinite-dimensional system. The above methods are not applicable to the performance (exponential decay rate or $L_2$-gain) analysis of the closed-loop infinite-dimensional systems.

In the recent papers [6] and [7] sampled-data control of 1-D diffusion PDEs under the spatially averaged and the point measurements respectively was studied. The results of [6], [7] were limited to the scalar 1-D case, whereas communication constraints (scheduling protocols) were not considered.

In the present paper we study, for the first time, a network-based $H_\infty$ filtering of distributed parameter systems. We consider a vector N-D semilinear diffusion PDE over a rectangular domain $\Omega$. We assume that there is available a large number of "point" spatial output measurements (e.g. of reactor temperature) so that the averaged measurements over the spatial domain $\Omega$ or over its closed sub-domains are known with sufficient accuracy. Note that a sensor cannot measure exactly in one point: the measuring device relies on some physical phenomenon and, in fact, the sensor measures an average over a certain region occupied by the measuring device. The measurements are sent over communication network to the observer in the discrete-time instances.

We suggest to divide the spatial domain into $N_s$ rectangular sub-domains, where sensing devices provide spatially averaged measurements. A larger $N_s$ allows to send a more accurate approximation of the "point" measurements that may improve the performance. However, due to communication constraints, an increase in $N_s$ (i.e. in the information to be transmitted) may enlarge the network-induced delays worsening the performance. Sufficient conditions for the internal exponential stability and $L_2$-gain analysis of the error system are derived in the framework of the time-delay approach to NCSs, where the variable in time sampling intervals and network-induced delays are taken into account.

A. Notation and Preliminaries

$R^N$ denotes the $N$-dimensional Euclidean space with the norm $| \cdot |$, $R^{N \times M}$ is the space of $N \times M$ real matrices, and the notation $P > 0$ with $P \in R^{N \times N}$ means that $P$ is symmetric and positive definite. The symmetric elements of the symmetric matrix will be denoted by $\ast$. If $A \in R^{N \times N}$ and $B \in R^{P \times q}$, then the Kronecker product $A \otimes B$ is the...
np × mq block matrix:

\[
A \otimes B = \begin{bmatrix}
  a_{11}B & \cdots & a_{1n}B \\
  \vdots & \ddots & \vdots \\
  a_{n1}B & \cdots & a_{nn}B
\end{bmatrix}.
\]

Continuous functions (continuously differentiable) in all arguments, are referred to as of class \(C^1\). \(L_2(\Omega)\) is the Hilbert space of square integrable functions \(f : \Omega \to \mathbb{R}^q\), where \(\Omega \subset \mathbb{R}^N\), with the norm \(\|f\|_{L_2} = \sqrt{\int_{\Omega} |f(x)|^2 \, dx}\). Let \(\partial \Omega\) be the boundary of \(\Omega\). \(L_2(0, \infty; L_2(\Omega))\) is the Hilbert space of square integrable functions \(w : (0, \infty) \to L_2(\Omega)\) with the norm \(\|w\|_{L_2(0, \infty; L_2(\Omega))}^2 = \int_0^\infty \int_{\Omega} |w(x,t)|^2 \, dx \, dt < \infty\).

For the diffusion process \(z(x,t) = [z^1(x,t), \ldots, z^M(x,t)]^T \in \mathbb{R}^M\) with \(z^m : \Omega \to \mathbb{R}\) \((m=1, \ldots, M)\) denote \(z^m_\tau = \left[\frac{\partial z^m}{\partial \tau}, \ldots, \frac{\partial z^m}{\partial \tau N}\right], \nabla_x z^m = (z^m_\tau)^T\) and \(\nabla_x z^m = \text{col}\{\nabla_x z^m_1, \ldots, \nabla_x z^m_N\} \in \mathbb{R}^{NM}\).

\(H^1(\Omega)\) is the Sobolev space of absolutely continuous functions \(z : \Omega \to \mathbb{R}^M\) with the square integrable \(\nabla_x z\).

\(\text{MATI}\) is the Maximum Allowable (network-induced) Delay.

\(\text{MAD}\) is the Maximum Allowable Transmission Interval.

\(\nabla\) is the gradient.

\(\delta\) is the diameter of \(\Omega\).

\(P\) is Poincaré’s constant.

\(W^2\) is the Sobolev norm.

\(\omega\) is the measurement noise.

\(\nu\) is the noise.

\(\theta\) is the disturbance.

\(\xi\) is the sensor measurement.

\(\delta\) is the boundary of \(\Omega\).

\(\xi\) is the boundary condition.

\(\tau\) is the time.

\(\Delta\) is the Laplacian.

\(D_{\Omega}\) is the diffusion term.

\(\hat{z}\) is the estimate of \(z\).

\(\bar{u}\) is the control input.

\(\xi\) is the measurement.

\(\omega\) is the noise.

\(\nu\) is the noise.

\(\theta\) is the disturbance.
$s_k$ denote the unbounded monotonously increasing sequence of sampling instances, i.e.

$$0 = s_0 < s_1 < ... < s_k < ..., \text{ } k \in \mathcal{N}, \lim_{k \to \infty} s_k = \infty.$$  

At each sampling instant $s_k$ the discrete-time measurements $y_{i,k} = y_i(s_k), i = 1, 2, ..., N_s$ are transmitted via the network.

This leads to the constrained data exchange expressed as

$$y_{i,k} = \frac{\int_{\Omega} C(z(x,t)) dx}{A_1} + w_{i,k}, \text{ } i = 1, 2, ..., N_s, k \in \mathcal{N},$$  

(10)

where $w_{i,k} = w_i(t_k)$ is an additive measurement disturbance (see Fig. 1). Denote

$$w_0(x,t) = w_{i,k}, x \in \Omega_i, t \in [t_k, t_{k+1}), i = 1, ..., N_s, k \in \mathcal{N}.$$  

(11)

We suppose that the transmission of the information over the network is subject to a variable and bounded communication delay $h_k \leq MAD$. Then $t_k = s_k + h_k$ is the updating instant time of the observer. We do not restrict the network-induced delay to be small with $t_k = s_k + h_k < s_{k+1}$, i.e. $h_k < s_{k+1} - s_k$. We assume the following:

**A1.** The time span between the most recent updating and the oldest sampling instant is bounded

$$t_{k+1} - t_k + h_k = s_{k+1} - s_k + h_k + h_k \leq \text{MATI} + \text{MAD} \triangleq \tau_M.$$  

(12)

**A2.** The measurements are sent with the time-stamps. The assumption A2 means that $s_k = t_k - h_k$ is known on the observer side. The latter allows to use the Lyapunov type observer of the form

$$\dot{z}(x,t) = \Delta_D e(x,t) - \beta \nabla_x e(x,t) + A \hat{e}(x,t) + \phi(z(x,t), x, t),$$  

for $t \in [0, t_{i-1})$,

$$\dot{z}(x,t) = \Delta_D e(x,t) - \beta \nabla_x e(x,t) + A \hat{e}(x,t) + \phi(z(x,t) \int C\hat{z}(x,s_k) ds_k \Delta_i)$$

$$+ K_o [y_{i,k} - \frac{\int_{\Omega} C\hat{z}(x,s_k) ds_k}{A_1}], \text{ } t \in [t_k, t_{k+1}), x \in \Omega_i, i = 1, ..., N_s, k \in \mathcal{N},$$  

(13)

with $\hat{z}(x,t) \in R^M$ and a constant observer gain $K_o$.

The observer dynamics is subject to the same boundary conditions as the state dynamics: $\dot{e}(x,t)|_{\partial \Omega} = 0$ for (3), (7) or $\hat{z}_t(x,t) \cdot n|_{\partial \Omega} = 0$ for (3), (8). By using the step method (i.e. considering $t \in [0, t_0), t \in [t_0, t_1), ...$) and applying the arguments of [6], the strong solutions of (13) under the corresponding boundary conditions initialized with $\hat{z}(\cdot, 0) \in H^1(\Omega)$ that satisfy the boundary conditions exist. Moreover, these solutions are continuatable for $t \geq 0$.

Let $e(x,t) = z(x,t) - \hat{z}(x,t)$ be the estimation error. Then the error dynamics is governed by

$$e_t(x,t) = \Delta_D e(x,t) - \beta \nabla_x e(x,t) + A e(x,t) + \phi'(e(x,t) + B_1 w(x,t),$$  

for $t \in [0, t_{i-1})$, $e_t(x,t) = \Delta_D e(x,t) - \beta \nabla_x e(x,t) + A e(x,t) + \phi'(e(x,t)$$  

$$- K_o [\int_{\Omega} C e(x, s_k) ds_k A_1] - K_o w_0(x,t) + B_1 w(x,t),$$  

(14)

$x \in \Omega_i, i = 1, 2, ..., N_s$,

where $\phi' \triangleq \int_0^1 \phi(\hat{z}(x,t) + \alpha e(x,t), x, t) d\alpha$. Due to (6), by applying Jensen’s inequality we obtain

$$\phi'^T \phi' \leq \int_0^1 \phi'^T (\hat{z} + \alpha e(x,t), x, t) d\alpha \leq Q$$  

(15)

for all $\hat{z}, e, x, t$. The boundary conditions for the error dynamics are

$$e(x,t)|_{\partial \Omega} = 0$$  

(16)

for (3) under the Dirichlet boundary conditions (7) and

$$e(x,t) \cdot n|_{\partial \Omega} = 0$$  

(17)

for (3) under the Neumann boundary conditions (8).

Following the time-delay approach to sampled-data control [9], denote

$$\tau(t) = t - t_k + h_k, t \in [t_k, t_{k+1}), i = 1, 2, ..., N_s, k \in \mathcal{N}.$$  

(18)

Then $t_{k+1} - t_k + h_k \leq \tau(t)$ and, due to (12),

$$\tau(t) \leq t_{k+1} - t_k + h_k \leq \tau_M, i = 1, 2, ..., N_s.$$  

Similar to [6], we shall use the elementary relation

$$\int_{t_{i-1}}^{t_i} e(x, t) - f_i(x, t) - \rho_i, x \in \Omega_i,$$

where

$$f_i(x, t) \triangleq \int_{t_{i-1}}^{t_i} \frac{e(x, t) - e(x, t)}{\Delta_i} - \rho_i \Delta \int_{t_{i-1}}^{t_i} \frac{e(x, t) ds_k}{\Delta_i}.$$  

(19)

Hence, the error system can be presented as

$$e_t(x,t) = \Delta_D e(x,t) - \beta \nabla_x e(x,t) + A e(x,t) + \phi'(e(x,t) + B_1 w(x,t)$$

$$+ K_o C p_i + B_1 w(x,t) - K_o w_0(x,t),$$  

(19)

$t \geq t_0, x \in \Omega_i, i = 1, ..., N_s$.

The initial condition $e(x,t)(t \in [0, t_0])$ for (19) is defined as a strong solution of (14), where $e(\cdot, 0) \in H^1(\Omega)$ satisfies the boundary conditions.

Since $f_{i_{N_s}}(x,t) = e_x(x,t)$ and $\int_{\Omega_i} f_i(x,t) dx = 0$, Poincare's inequality (1) implies

$$\int_{\Omega_i} \left| \nabla e^m(x,t) \right|^2 - \frac{\pi^2}{\Delta_i} \left| f_i^m(x,t) \right|^2 \right] dx \geq 0, t \geq t_{N_s-1},$$  

(20)
Note that the Lyapunov-based analysis of (19) under the corresponding boundary conditions in the case of scalar $z$ and $x$ with $s_k = t_k$ and $h_k = 0$ was considered in [6], where $\rho_t$ was multiplied by $1/\rho_t$.

For the system (19), we choose the Lyapunov-Krasovskii functional of the form
\[
V(t) = V_P + V_S + V_R,
\]
\[
V_P = \int_\Omega e^T(x,t)P_1e(x,t)dx,
\]
\[
V_S = \frac{1}{M} \sum_{m=1}^{M} \int_\Omega P_3^{m}e^m(x,t)D^m(x)\nabla_x e^m(x,t)dx,
\]
\[
V_R = \tau M \int_0^t \int_{-\tau M}^t \int_{+\theta}^t e^{2\alpha(s-t)}e^T(x,s)Se(x,s)dsdx,
\]
where $e(x,t) \equiv e(x,0)$ for $t < 0$. Our objective is to find a constant gain $K_\alpha$ that internally exponentially stabilizes the error system, i.e., exponentially stabilizes the disturbance-free system in the sense that the following holds:
\[
V(t) = e^{-2\alpha(t-t_0)}V(t_0), \quad t \geq t_{N_*} - 1,
\]
where $\alpha > 0$ is the decay rate. While internally stabilizing the parabolic error process, the influence of the admissible external disturbances on the controlled output
\[
\zeta(x,t) = C_1e(x,t)
\]
with a constant matrix $C_1 \in \mathbb{R}^{K \times M}$ is to be attenuated.

The following $H_\infty$ filtering problem is thus under study. Given $\gamma > 0$, it is required to find an observer (13) that internally exponentially stabilizes the error dynamics (19) and leads to a negative performance index
\[
J(T) = \int_0^T \int_\Omega \left[ |\zeta(x,t)|^2 - \gamma^2 \|w(x,t)\|^2 + |w_0(x,t)|^2 \right] dxdt < V(t_{N_*)} - 1 \quad \forall T > t_0
\]
for all admissible disturbances such that
\[
\int_\Omega \|w(x,t)\|^2 + |w_0(x,t)|^2 dx > 0, \quad t > t_0.
\]
Then, it is said that the error dynamics (19) has an $L_2$-gain less than $\gamma$.

III. MAIN RESULTS: $L_2$-GAIN ANALYSIS AND DESIGN

In order to solve the problem, we will derive sufficient conditions for the following dissipative inequality
\[
W(t) \geq \dot{V}(t) + 2\alpha V(t) + \sum_{i=1}^{N} \int_\Omega \{ |\zeta(x,t)|^2 - \gamma^2 \|w(x,t)\|^2 + |w_0(x,t)|^2 \} dxdt < 0, \quad \alpha > 0, \quad t \geq t_0
\]
to hold along the trajectories of (19) with the corresponding boundary conditions provided (24) is valid. The integration of (25) in $t$ from $t_0$ to $T$ would yield (23) since $V \geq 0$. For the unperturbed system (19), (25) implies $V(t) + 2\alpha V(t) \leq 0$ and, thus, the exponential stability of (19) with the decay rate $\alpha$.

**Theorem 1:** Given positive scalars $N, \delta, \gamma, \alpha, \tau_M$ and a matrix $K_\alpha \in \mathbb{R}^{M \times M}$, let there exist a matrix $G \in \mathbb{R}^{M \times M}$, positive $M \times M$-matrices $P_1, S$ and $R_M$, diagonal $M \times M$-matrices $P_3 > 0$ and $P_2$, and scalars $\lambda_0 > 0, \lambda_1 > 0 (j = 1, 2)$ such that the Neumann boundary conditions for the following LMIs are feasible:
\[
\Xi \triangleq \left[ \begin{array}{cc} R & G \\ G^T & \gamma I \end{array} \right] > 0, \quad \Phi_\gamma < 0,
\]
where
\[
\Phi_\gamma = \left[ \begin{array}{cccc} \Psi_1 & \cdots & \Psi_M \\ \Psi_2 & \cdots & \Psi_M \\ \vdots & \cdots & \vdots \\ \Psi_1 & \cdots & \Psi_M \end{array} \right] < 0
\]
and
\[
\Psi_m = \left[ \begin{array}{cccc} \Phi_{11} & \cdots & \Phi_{1M} & \Phi_{21} \\ \Phi_{22} & \cdots & \Phi_{2M} & \Phi_{31} \\ \vdots & \cdots & \vdots & \vdots \\ \Phi_{M1} & \cdots & \Phi_{MM} & \Phi_{M2} \end{array} \right] < 0
\]
with positive matrices $P_1, S, R_M \in \mathbb{R}^{M \times M}$ and positive diagonal $M \times M$-matrix $P_3 = \text{diag}(p^1_3, \ldots, p^M_3)$. This $V$ extends the construction of [7] to N-D case. Following [12], in order to guarantee that $V(t)$ is defined for all $t \geq t_0$ we set $e(x,t) \equiv e(x,0)$ for $t < 0$.

Here
\[
\Phi_{11} = 2\alpha P_1 + P_2(A - K_\alpha C) + (A - K_\alpha C)^T P_2 + S + \lambda_0 \sum_{k=1}^{K} p^k I_M + \lambda_0 Q + C^T C,
\]
\[
\Phi_{12} = P_1 - P_2 - P_3(A - K_\alpha C),
\]
\[
\Phi_{13} = \Phi_{14} - P_3 K_\alpha C,
\]
\[
\Phi_{16} = (R - G)e^{-2\alpha t_M} - \lambda_2 I_M,
\]
\[
\Phi_{22} = R^2 - 2P_3,
\]
\[
\Phi_{23} = R^2 - P_3 K_\alpha C,
\]
\[
\Phi_{25} = -P_3 \beta,
\]
\[
\Phi_{28} = P_3,
\]
\[
\Phi_5 = -20(2 - P_0 - \beta I_M) \| \alpha \| I_M + |\alpha_0 + \lambda_1| I_{MN},
\]
\[
\Phi_6 = (G + G^T - 2R^2)e^{-2\alpha t_M} + \lambda_0 I_M,
\]
\[
\Phi_7 = (R - G)e^{-2\alpha t_M} + \Phi_7 = -Re^{-2\alpha t_M} - Se^{-2\alpha t_M}.
\]

Then the error system (19) under the Dirichlet (16) or under the Neumann (17) boundary conditions is internally exponentially stable with the decay rate $\alpha$ and has $L_2$-gain less than $\gamma$. Moreover, if the above conditions are feasible with $\alpha = 0$, then (19) is internally exponentially stable with a small enough decay rate and has $L_2$-gain less than $\gamma$.

See Appendix for the proof.

**Remark 1:** Note that $\delta$ appears only in $\Phi_{33} = -\lambda_2^2 I_M$ of $\Phi_\gamma$, meaning that a smaller $\delta$ enforces $\tau_M$ that preserves the $H_\infty$ performance. Therefore, given $N_*$ one has to choose such a division of $\Omega$ that minimizes the maximum diameter of the resulting sub-domains. This choice enlarges $MATI = \tau_M - M AD$.

**Remark 2:** Comparatively to [6] and [7], an improved technique (based on S-procedure) is presented (see Appendix) leading to less restrictive LMIs under the Dirichlet than under the Neumann boundary conditions. The LMIs are less restrictive due to the negative term $-\lambda_0 \sum_{k=1}^{K} p^k I_M$ in $\Phi_{11}$ of (27).
Consider $H_\infty$ filtering of the scalar 2-D PDE (3) with the controlled output (22) and the measurements (10), where $d_0 = 10^{-4}, A = \beta = \phi = 0, \Omega = [0, 0.1] \times [0, 0.06]$ and $C_1 = B_1 = C = 1$ under the Dirichlet boundary conditions.

This PDE with $B_1 = 0$ was considered in [4]. We take $N_s$ as in the Table 1 and choose such a division of $\Omega$ into $N_s$ equal rectangles that corresponds to the minimum diameter $\delta$ (see Table 1). Thus, for $N_s = 2$ the division of the side $[0, 0.1]$ into two subintervals (i.e., $N_s = 2 \times 1$) leads to a smaller $\delta^2 = 0.055^2 + 0.06^2 = 0.0061$ than the division of the side $[0, 0.06]$ into two subintervals ($N_s = 1 \times 2$) with $\delta^2 = 0.11^2 + 0.03^2 = 0.019$. Using Theorem 1 with $K_0 = 0.2$, $\alpha = 0$ and $\gamma = 5.3$ we find the maximum values of $\tau_M$ that guarantee $J(T) < 0$ (see Table 1). Note that under the Neumann boundary conditions, where the LMIs of Theorem 1 are used with $\lambda_0 = 0$, the resulting values of $\gamma$ (for the same values of $\tau_M$ and $\delta$) are essentially larger: e.g. for $N_s = 2 \times 1$ and $\tau_M = 2.5$ the resulting $\gamma = 10.1$ (instead of 5.3 in Table 1).

For large MAD=3.5, only the division into $N_s = 6$ subdomains preserves the error performance.

Simulation of $J(T)$ corresponding to the error system confirm the theoretical results.

V. Conclusion

$H_\infty$ filter has been designed for convection-diffusion PDEs over rectangular N-D domain $\Omega$ in the situation, where distributed in space measurements are sent to observer through communication network. The objective is to enlarge the sampling time intervals, while preserving a satisfactory error system performance in spite of varying sampling and network-induced delays. We have suggested to divide $\Omega$ into a finite number $N_s$ of rectangular sub-domains, where sensing devices provide spatially averaged measurements. Given $N_s$, we have found that the division that minimizes the maximum diameter of the resulting sub-domains is advantageous leading to larger sampling intervals. LMIs have been derived for the bounds on the network-induced delays, sampling time intervals and diameters of measurement sub-domains that guarantee a desired performance of the estimation error.

APPENDIX

We will derive LMI conditions for $W(t) < 0 \ (t \geq t_{N_s-1})$ via the descriptor method [5], where

$$0 \cong 2 \int_\Omega [e^T(x, t)P_2 + e^T_t(x, t)P_3] - e_t(x, t) + \Delta D e(x, t) \notag$$

$$+ \Delta e(x, t) - \beta \nabla e(x, t) + (\phi' - K_0 C) e(x, t) + B_1 w(x, t)] dx$$

$$+ 2 \sum_{i=1}^N \int_\Omega [e^T(x, t)P_2 + e^T_t(x, t)P_3] K_0 C [f_i(x, t)$$

$$+ p_i - w_0(x, t)] dx$$

with some free diagonal matrix $P_2 = \text{diag}\{p_1^2, ..., p_M^2\}$. Taking into account the boundary conditions, by Green’s formula we obtain

$$2 \int_\Omega e^T_t(x, t)P_3 \Delta p e(x, t) dx$$

$$= -2 \sum_{m=1}^M e^T_m \int_\Omega e^T_m(x, t)(\Delta p e^T_m(x, t)) dx$$

$$- 2 \int_\Omega e^T_t(x, t)P_2 \Delta D e(x, t) dx$$

We have

$$\dot{V}_P(t) + 2aV_P(t) = 2 \int_\Omega e^T(x, t) P_1 e(x, t) dx$$

$$+ 2 \sum_{m=1}^M \int_\Omega e^T(x, t) P_2 e^T_m(x, t) dx$$

$$+ 2 \sum_{m=1}^M \int_\Omega e^T(x, t) P_3$$

$$+ \Delta e^T(x, t) [A + \phi' - K_0 C] e(x, t) + B_1 w(x, t)$$

$$+ K_0 C [f_i(x, t) + p_i - w_0(x, t)]] dx.$$

Then adding (29) to (31) and taking into account (30) we arrive at

$$\dot{V}_P(t) + 2aV_P(t) = 2 \int_\Omega e^T(x, t) P_1 e(x, t) dx$$

$$+ 2 \sum_{m=1}^M \int_\Omega e^T(x, t) P_2 e^T_m(x, t) dx$$

$$- \sum_{m=1}^M (p_m^2 - \alpha p_m^2) e^T_m(x, t) D^m e(x, t)$$

$$- \beta \nabla e(x, t) + [A + \phi' - K_0 C] e(x, t) + B_1 w(x, t)$$

$$+ K_0 C [f_i(x, t) + p_i - w_0(x, t)]] dx.$$}

The feasibility of $\Phi_\gamma < 0$ implies that $\Phi_{55} > 0$ and, thus, $p_2^m - \alpha p_3^m > 0$. Hence, due to (4),

$$-2 \sum_{m=1}^M (p_m^2 - \alpha p_m^2) \int_\Omega e^T_m(x, t) D^m e(x, t) dx$$

$$\leq -2 \int_\Omega \nabla e^T(x, t) [D_0 (P_2 - \alpha P_3) \otimes I_N] \nabla e(x, t) dx.$$
\[ e^T(x, t - \tau_M) \]. Applying Jensen's inequality and convex analysis of [14], we obtain
\[ -\tau_M \int_t^{t - \tau} e^T(x, s) Re_s(x, s) ds + \int_0^{t - \tau} e^T(x, s) Re_s(x, s) ds \leq -\xi^T(x, t) \text{diag} \left( \frac{1}{\alpha_0} R, \frac{1}{\alpha_2} R \right) \xi(x, t) \leq -\xi^T(x, t) \Xi \xi(x, t), \]
where \( \Xi > 0 \) due to (26). Then
\[
\dot{V}_R(t) + 2\alpha V_R \leq \tau^2 \int_0^{t} e^T(x, t) Re_t(x, t) dx - e^{2\alpha \tau_M} \int_0^{t} \eta_0^T \left[ \begin{array}{cc} R & G - R \\ G^T - R & G^T - G \end{array} \right] \eta_0 dx,
\]
where
\[
\eta_0 = \left[ e^T(x, t) \ e^T(x, t - \tau(t)) \ e^T(x, t - \tau_M) \right].
\]
By Jensen’s Inequality for \( i = 1, 2, \ldots, N_s \)
\[
\int_0^{t} \left( |e(x, t) - e(x, t - \tau(t))| \right) dx \geq 0,
\]
(33)
where \( i = 1, 2, \ldots, N_s \). Summation in (20) leads to
\[
\sum_{i=1}^{N_s} \int_0^{2\Delta_i} |\eta_0(x) - e(x, t - \tau(t))| dx |^2 \geq 0.
\]
(34)
Similarly, under the Dirichlet boundary conditions, the Wirtinger inequality (2) implies
\[
\sum_{i=1}^{N_s} \int_0^{2\Delta_i} \left| \nabla e(x, t) \right|^2 - \sum_{i=1}^{N_s} |f_i(x, t)|^2 dx \geq 0.
\]
(35)
Taking into account (33)-(35) and (15) and applying the S-procedure, we add to \( \dot{V} + 2\alpha V \) the left-hand sides of
\[
\lambda_0 \left\{ \sum_{i=1}^{N_s} \int_0^{2\Delta_i} \left| \nabla e(x, t) \right|^2 - \sum_{i=1}^{N_s} |f_i(x, t)|^2 dx \right\} \geq 0,
\]
\[
\lambda_1 \left\{ \sum_{i=1}^{N_s} \int_0^{2\Delta_i} \left| \nabla e(x, t) \right|^2 - \sum_{i=1}^{N_s} |f_i(x, t)|^2 dx \right\} \geq 0,
\]
\[
\lambda_2 \left\{ \sum_{i=1}^{N_s} \int_0^{2\Delta_i} \left| e(x, t) - e(x, t - \tau(t)) \right|^2 - |\eta_0|^2 dx \right\} \geq 0,
\]
\[
\lambda_Q \left\{ \sum_{i=1}^{N_s} \int_0^{2\Delta_i} \left| e^T(x, t) Q e(x, t) - |\phi_1 e(x, t)|^2 \right| dx \right\} \geq 0,
\]
(36)
where \( \lambda_j \geq 0 (j = 0, 1, 2) \) and \( \lambda_Q > 0 \) are some constants. Finally, from (32)-(36) it follows that (26) yields (25) provided (24) holds, where
\[
W(t) \leq \sum_{i=1}^{N_s} \int_0^{2\Delta_i} \eta_0^T \Phi_i \eta_0 dx < 0, \quad t \geq t_{N_s - 1},
\]
\[
\eta_0 = \text{col} \left\{ e(x, t), e(x, t), f_i(x, t), \rho_0 \nabla e(x, t), e(x, t - \tau(t)), e(x, t - \tau_M), \phi_1 e(x, t), w(x, t), w_0(x, t) \right\}.
\]

REFERENCES