Compositional Approximations of Interconnected Stochastic Hybrid Systems

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Abstract—This paper provides a compositional approach for approximations of the interconnection of a class of stochastic hybrid systems including both jump linear stochastic systems and linear stochastic hybrid automata. The approximation is based on the recently developed notions of stochastic (bi)simulation functions using which one can quantify the error between original stochastic hybrid systems and their abstractions. Given stochastic (bi)simulation functions between stochastic hybrid subsystems and their corresponding approximations, we provide a stochastic (bi)simulation function between an interconnection of those stochastic hybrid subsystems and that of their corresponding approximations. Consequently, one can leverage the proposed results here to solve particularly safety/reachability problems over the interconnection of abstract systems and then carry the results over the interconnection of concrete ones. We illustrate the effectiveness of the proposed results here by computing a stochastic (bi)simulation function between an interconnection of two identical jump linear stochastic subsystems and that of their corresponding approximations by just using the stochastic (bi)simulation function between one of the jump linear stochastic subsystems and its corresponding approximate abstraction.

I. INTRODUCTION

Stochastic hybrid systems are a general class of dynamical systems consisting of continuous and discrete dynamics subject to probabilistic noise and events. In the past few years, these systems have become ubiquitous in a variety of fields due to providing a rigorous and general framework for modeling of many safety critical applications. Examples of those applications include biochemistry [1], air traffic control [2], systems biology [3], and communication networks [4], that can be modeled and treated as stochastic hybrid systems.

Reachability and safety are two fundamental concepts in the study of stochastic hybrid systems that have received significant attentions in the last few years because of their applications in real life systems [5]. There exist some results in the literature such as [6], [7] providing methodologies solving reachability problems, safety ones, or their combinations. Unfortunately, the computational complexity of the existing techniques solving those problems scales exponentially with the dimension of the continuous state space making them intractable for large-scale stochastic hybrid systems. A promising direction to overcome this complexity is to abstract the original concrete system by a simpler one (lower dimension) using an appropriate relation quantifying the error between them. By taking into account the quantified error, one can perform the analysis and synthesis over the simpler system and then carry the results over the concrete one.

The recent work in [8] provides a notion of stochastic (bi)simulation function providing a metric which quantifies the error between the original stochastic hybrid systems and their approximations. Particularly, the authors in [8] develop a theory of approximate (bi)simulation for a class of stochastic hybrid automata whose continuous dynamics are modeled by continuous-time stochastic differential equations and the switches are modeled as Poisson processes. In this work, we provide a compositional technique for approximation of the interconnection of the same class of stochastic hybrid automata. Given stochastic (bi)simulation functions between each stochastic hybrid subsystem and its corresponding approximation, we provide a stochastic (bi)simulation function between the interconnection of stochastic hybrid subsystems and that of their corresponding approximations. Therefore, one can quantify the error between the interconnection of stochastic hybrid subsystems and that of their corresponding abstractions by just knowing the error between each subsystem and its corresponding abstraction. As a consequence, one can leverage the proposed results here to solve particularly safety/reachability problems over the interconnection of abstract systems and then carry the results over the interconnection of concrete ones.

We illustrate the results of this paper by computing a stochastic (bi)simulation function between an interconnection of two identical jump linear stochastic subsystems (3rd order each) and that of their corresponding abstractions (1st order each) by using just the stochastic (bi)simulation function between one of the subsystems and its corresponding approximation.

II. STOCHASTIC HYBRID SYSTEMS

A. Notations

The identity map on a set $A$ is denoted by $1_A$. The symbols $\mathbb{N}$, $\mathbb{R}$, $\mathbb{R}^+$, and $\mathbb{R}^*_0$ denote the set of natural, real, positive, and nonnegative real numbers, respectively. Given a vector $x \in \mathbb{R}^n$, we denote by $x_i$ the $i$-th element of $x$, and by $\|x\|$ the Euclidean norm of $x$, namely, $\|x\| = \sqrt{x_1^2 + x_2^2 + \ldots + x_n^2}$. We denote by $I_n$ and $0_n$ the identity matrix and zero vector in $\mathbb{R}^{n \times n}$ and $\mathbb{R}^n$, respectively. For any $x, y \in \mathbb{R}^n$, the relation $x \leq y$ is defined by $x_i \leq y_i$ for all $i \in \{1, \ldots, n\}$. The relations $<$, $>$, and $\geq$ are defined in the same manner for vectors. Given a matrix $P = \{p_{ij}\} \in \mathbb{R}^{n \times n}$, we denote by $\text{Tr}(P) = \sum_{i=1}^n p_{ii}$ the trace of $P$ and by $\hat{p}(P)$ the spectral radius of $P$, namely, $\hat{p}(P) = \max_i \|X_i\|$, where $X_i, i \in \{1, \ldots, N\}$, are eigenvalues of $P$. We denote by $\text{diag}(a_1, \ldots, a_N)$ a diagonal matrix in $\mathbb{R}^{N \times N}$ whose diagonal entries starting from the upper left corner are $a_1, \ldots, a_N$. Similarly, we denote by $\text{diag}(A_1, \ldots, A_N)$ a block diagonal matrix whose diagonal entries starting...
from the upper left corner are matrices \( A_1, \ldots, A_N \). Given functions \( f_i: X_i \to Y_i \), for any \( i \in \{ 1, \ldots, N \} \), their Cartesian product \( \prod_{i=1}^N f_i: \prod_{i=1}^N X_i \to \prod_{i=1}^N Y_i \) is defined as \( (f_1(x_1), \ldots, x_N) = (f_1(x_1)(x_1), \ldots, f_N(x_N)) \).

Given a measurable function \( f: \mathbb{R}_+^n \to \mathbb{R}^n \), the (essential) supremum of \( f \) is denoted by \( \| f \|_\infty \); we recall that \( \| f \|_\infty := \text{ess sup}_{t \geq 0} \| f(t) \| \). A continuous function \( \gamma: \mathbb{R}_+^n \to \mathbb{R}_+^n \) is said to belong to class \( K \) if it is strictly increasing and \( \gamma(0) = 0 \); \( \gamma \) is said to belong to class \( K_\infty \) if \( \gamma \in K \) and \( \gamma(s) \to \infty \) as \( s \to \infty \).

B. Stochastic hybrid systems

Let \( (\Omega, \mathcal{F}, \mathbb{P}) \) be a probability space endowed with a filtration \( \mathcal{F} = (\mathcal{F}_s)_{s \geq 0} \) satisfying the usual conditions of completeness and right continuity [9, p. 48]. Let \( (W_s)_{s \geq 0} \) be a \( p \)-dimensional F-Brownian motion and \( (P_s)_{s \geq 0} \) be a \( q \)-dimensional F-Poisson process. We assume that the Poisson process and the Brownian motion are independent of each other. The Poisson process \( P_s := [P^1_s; \ldots; P^q_s] \) model \( q \) kinds of events whose occurrences are assumed to be independent of each other.

**Definition 2.1:** The class of stochastic hybrid systems with which we deal in this paper is the tuple \( \Sigma = (X, U, \mathcal{U}, f, \sigma, r, Y, h) \), where
- \( X \subseteq \mathbb{R}^n \) is the state set;
- \( U \subseteq \mathbb{R}^n \) is the input set;
- \( \mathcal{U} \) is a subset of the set of all measurable functions of time intervals of the form \([0, \infty]\) to \( U \);
- \( f: X \times U \to \mathbb{R}^n \) is the drift term which is globally Lipschitz continuous: there exist constants \( L_x, L_u \in \mathbb{R}^+ \) such that: \( \| f(x, u) - f(x', u') \| \leq L_x \| x - x' \| + L_u \| u - u' \| \) for all \( x, x' \in X \) and all \( u, u' \in U \);
- \( \sigma: \mathbb{R}^n \to \mathbb{R}^{n \times p} \) is the diffusion term which is globally Lipschitz continuous;
- \( r: \mathbb{R}^n \to \mathbb{R}^{n \times q} \) is the reset function which is globally Lipschitz continuous;
- \( Y \subseteq \mathbb{R}^q \) is the output set;
- \( h: X \to Y \) is the output map.

A stochastic hybrid system \( \Sigma \) satisfies

\[
\Sigma: \left\{ \begin{array}{l}
d\xi(t) = f(\xi(t), u(t)) \, dt + \sigma(\xi(t)) \, dW_t + r(\xi(t)) \, dP_t, \\
\eta(t) = h(\xi(t)), 
\end{array} \right.
\]

**III. Stochastic (Bi)Simulation Function**

We recall the notion of stochastic (bi)simulation function, introduced in [8], with a slight modification, which is useful to relate properties of stochastic hybrid systems to those of their abstractions.

**Definition 3.1:** Let \( \Sigma = (X, U, \mathcal{U}, f, \sigma, r, Y, h) \) and \( \Sigma = (\bar{X}, \bar{U}, \bar{f}, \bar{r}, \bar{Y}, \bar{h}) \) be two stochastic hybrid systems such that \( Y \) and \( \bar{Y} \) are of equal dimension. A continuous function \( \phi: X \times \bar{X} \to \mathbb{R}_+^0 \cup \{ +\infty \} \) is a stochastic simulation function from \( \Sigma \) to \( \bar{\Sigma} \) if:

(i) for every \( x \in X \) and \( \bar{x} \in \bar{X} \), \( \phi(x, \bar{x}) \geq \alpha(\| h(x) - \bar{h}(\bar{x}) \|) \) for some \( \alpha \in K_\infty \);

(ii) for any \( v \in \mathcal{U} \), there exists \( \bar{v} \in \bar{U} \) such that the stochastic process \( \phi(\xi_{av}(t), \bar{X}_{\bar{av}}(t)) \) is a supermartingale [11, Appendix C] for any random variable \( a \) and \( \bar{v} \) that are measurable in \( F_0 \).

Stochastic hybrid system \( \Sigma \) is simulated by \( \bar{\Sigma} \), denoted by \( \Sigma \preceq \bar{\Sigma} \), if there exists a stochastic simulation function from \( \Sigma \) to \( \bar{\Sigma} \).

The following theorem, inspired by Theorem 3 in [8], clarifies the significance of stochastic simulation functions.

**Theorem 3.2:** Let \( \Sigma \) and \( \bar{\Sigma} \) be two stochastic hybrid systems and \( \phi \) be a stochastic simulation function from \( \Sigma \) to \( \bar{\Sigma} \). For any solution process \( \xi_{av} \) of \( \Sigma \), there exists a solution process \( \bar{\xi}_{\bar{av}} \) of \( \bar{\Sigma} \) such that

\[
P \left( \sup_{t \in \mathbb{R}_+^0} \| \bar{h}(\xi_{av}(t)) - \bar{h}(\bar{X}_{\bar{av}}(t)) \| > \varepsilon \mid (a, \bar{v}) \right) \leq \frac{\phi(a, \bar{v})}{\alpha(\varepsilon)}.
\]

**Proof:** Following the definition of stochastic simulation function in Definition 3.1, for any input signal \( v \in \mathcal{U} \), there exists an input signal \( \bar{v} \in \bar{U} \) such that \( \phi(\xi_{av}(t), \bar{X}_{\bar{av}}(t)) \) is a nonnegative supermartingale. As a result, we have the following chain of inequalities:

\[
P \left( \sup_{t \in \mathbb{R}_+^0} \| \bar{h}(\xi_{av}(t)) - \bar{h}(\bar{X}_{\bar{av}}(t)) \| > \varepsilon \mid (a, \bar{v}) \right) = 1
\]

\[
P \left( \sup_{t \in \mathbb{R}_+^0} \| \bar{h}(\xi_{av}(t)) - \bar{h}(\bar{X}_{\bar{av}}(t)) \| > \varepsilon \mid (a, \bar{v}) \right) \leq \frac{\phi(a, \bar{v})}{\alpha(\varepsilon)}
\]

where the last inequality is implied from \( \phi(\xi_{av}, \bar{X}_{\bar{av}}) \) being a nonnegative supermartingale and [12, Lemma 1].

A symmetric version of a stochastic simulation function is called a stochastic bisimulation function as defined next.

**Definition 3.3:** Let \( \Sigma = (X, U, \mathcal{U}, f, \sigma, r, Y, h) \) and \( \bar{\Sigma} = (\bar{X}, \bar{U}, \bar{f}, \bar{r}, \bar{Y}, \bar{h}) \) be two stochastic hybrid systems such that \( Y \) and \( \bar{Y} \) are of equal dimension. A continuous function \( \phi: X \times \bar{X} \to \mathbb{R}_+^0 \cup \{ +\infty \} \) is a stochastic bisimulation function from \( \Sigma \) to \( \bar{\Sigma} \) if it is both a stochastic simulation function from \( \Sigma \) to \( \bar{\Sigma} \) and from \( \bar{\Sigma} \) to \( \Sigma \).

\(^1\)Note that \( F_0 \) can be the trivial sigma-algebra as well, i.e., \( a \) and \( \bar{v} \) can be non-probabilistic initial conditions as well.
Stochastic hybrid system \( \Sigma \) is bisimilar to \( \bar{\Sigma} \), denoted by \( \Sigma \simeq_{S} \bar{\Sigma} \), if there exists a stochastic bisimulation function between \( \Sigma \) and \( \bar{\Sigma} \).

Similar result as the one of Theorem 3.2 can be established using stochastic bisimulation function as the following.

Corollary 3.4: Let \( \Sigma \) and \( \bar{\Sigma} \) be two stochastic hybrid systems and \( \phi \) be a stochastic bisimulation function between them. For any solution process \( \xi_{s} \) of \( \Sigma \), there exists a solution process \( \bar{\xi}_{s} \) of \( \bar{\Sigma} \) and vice versa, such that inequality (III.1) holds.

Proof: The proof is similar to the one of Theorem 3.2.

Given a complex stochastic hybrid system \( \Sigma \) and its abstraction \( \Sigma \), one can use the provided results in Theorem 3.2 and Corollary 3.4 to quantify the distance between those systems. Specifically, one can leverage the results in Theorem 3.2 to verify the original system \( \Sigma \) against some complex specifications, e.g. linear temporal logic (LTL), by just verifying its abstraction \( \bar{\Sigma} \) against the same specifications. For example, Theorem 7 in [8] explains how an upper bound of the unsafety risk of the complex system \( \Sigma \) can be computed by performing the risk calculation over an abstraction \( \bar{\Sigma} \) simulating \( \Sigma \). Moreover, one can use the results in Corollary 3.4 to synthesize control policies enforcing some complex specifications, e.g. LTL, on the original system \( \Sigma \) by just refining the corresponding control policies enforcing the same specifications on its abstraction \( \Sigma \).

The following theorem provides a sufficient condition for the construction of a stochastic (bi)simulation function.

Theorem 3.5: Let \( \Sigma = (X, U, \mathcal{U}, f, \sigma, r, Y, h) \) and \( \bar{\Sigma} = (\bar{X}, \bar{U}, \bar{\mathcal{U}}, \bar{f}, \bar{\sigma}, \bar{r}, \bar{Y}, \bar{h}) \) be two stochastic hybrid systems such that \( Y \) and \( \bar{Y} \) are of equal dimension and \( U = \bar{U} \).

Consider a function \( \phi : X \times \bar{X} \to \mathbb{R}_{+}^{n} \cup \{+\infty\} \) that is twice continuously differentiable. If condition (i) in Definition 3.1 is satisfied for some \( \alpha, \rho \in \mathbb{R}_{+}^{n} \) and there exist functions \( \alpha, \rho \in \mathbb{R}_{+}^{n} \) such that

\[
\text{(ii) for any } x \in X, \bar{x} \in \bar{X}, \text{ and for any } u \in U, \pi \in \bar{U},
\]

\[
\mathcal{L}^{u,\pi}_{\phi}(x, \bar{x}) := [\partial_{u}\phi] f(x, u) + \frac{1}{2} \text{tr} \left( \begin{bmatrix} \sigma(x) \sigma^{T}(x) \end{bmatrix} \begin{bmatrix} \partial_{u,\pi} \phi \partial_{u,\pi} \phi \end{bmatrix} \right) + \sum_{i=1}^{q} \lambda_{i} \left(\phi(x + r(x)e_{i}, \pi + \tau(y)e_{i}) - \phi(x, \bar{x})\right)
\]

\[
\leq -\alpha(\phi(x, \bar{x})) + \rho(||u - \pi||),
\]

where \( e_{i} \in \mathbb{R}^{n} \) denotes the vector with 1 in the \( i \)th coordinate and 0’s elsewhere, then \( \phi \) is a stochastic bisimulation function between \( \Sigma \) and \( \bar{\Sigma} \).

In the above theorem, \( \mathcal{L}^{u,\pi}_{\phi} \) is the infinitesimal generator of the process \( \zeta = [\xi, \xi] \) [11, Section 7.3] and the symbols \( \partial_{u} \) and \( \partial_{x,\pi} \) denote the first and the second order partial derivatives with respect to \( x \) and \( x, \pi \), respectively.

Proof: Condition (i) in Definition 3.1 is trivially satisfied. Condition (ii) can also be readily verified by using Dynkin’s formula [11], similar to the proof of Theorem 8 in [8].

We refer the interested readers to Lemma 9 in [8] providing equivalent conditions as (i) and (ii) in Theorem 3.5 for two JLSS and a quadratic function \( \phi \).

IV. COMPOSITIONAL APPROXIMATIONS FOR INTERCONNECTED SYSTEMS

First we provide a formal definition of interconnection between stochastic hybrid systems.

A. Interconnection

This subsection introduces the notion of interconnection between \( N \) stochastic hybrid subsystems similar to the one introduced in [13]. As an example, Figure 1 illustrates the interconnection of two subsystems.

Consider a complex stochastic hybrid system \( \Sigma \) composed of \( N \) stochastic hybrid subsystems \( \Sigma_{i} \), interconnected with each other. Here, we assume that any input value \( u_{i} \in U_{i} \), any output value \( y_{i} \in Y_{i} \), and correspondingly the output function \( h_{i} \) are decomposed to subvectors as depicted in Figure 2 and shown as the following:

\[
u_{i} = [w_{i1}; w_{i}], \quad \text{s.t. } u_{i} = [w_{i1}; \ldots; w_{i,i-1}; w_{i,i+1}; \ldots; w_{iN}],
\]

\[
y_{i} = h_{i}(x_{i}) = [h_{i}^{1}(x_{i}); h_{i}^{2}(x_{i})] = [y_{i}; z_{i}],
\]

\[
\text{s.t. } z_{i} = [z_{i1}; \ldots; z_{i,i-1}; z_{i,i+1}; \ldots; z_{iN}].
\]

Fig. 2. Input/output configuration of subsystem \( \Sigma_{i} \).

The values \( u_{i} \) and \( y_{i} \) are called internal values, used to construct interconnection between subsystems. On the other hand, the values \( u_{i} \) and \( y_{i} \) are called external values together with those of other subsystems construct the input/output configuration of the overall interconnected system \( \Sigma \). Note that the external values remain accessible even after interconnecting subsystems. Now we can formally define the interconnection between stochastic hybrid subsystems.

Definition 4.1: Consider \( N \) stochastic hybrid subsystems \( \Sigma_{i} = (X_{i}, U_{i}, \mathcal{U}_{i}, f_{i}, \sigma_{i}, r_{i}, Y_{i}, h_{i}), i \in \{1, \ldots, N\} \), whose input and output sets and output functions are decomposed as in (IV.1) and (IV.2): \( U_{i} = U_{i1} \times W_{i1}, Y_{i} = Y_{i1} \times Z_{i1}, \) and \( h_{i} = h_{i1}^{1} \times h_{i1}^{2} \) for some appropriate sets \( U_{i1}, W_{i1}, Y_{i1}, Z_{i1} \) and functions \( h_{i1}^{1}, h_{i1}^{2} \). Suppose the sizes of subvectors \( w_{i1} \) and \( z_{i1} \) are equal for all \( i \in \{1, \ldots, N\}, j \in \{1, \ldots, N\} \setminus \{i\} \). The interconnection of \( \Sigma_{i} \)’s, \( i \in \{1, \ldots, N\} \), is the stochastic
hybrid system $\Sigma = (X, U, \mathcal{U}, f, \sigma, r, Y, h)$, denoted by $\Sigma = \mathcal{I}(\Sigma_1, \ldots, \Sigma_N)$, where $X = \prod_{i=1}^{N} X_i$, $U = \prod_{i=1}^{N} U_i$, $\mathcal{U} = \{ u \in \prod_{i=1}^{N} U_i \mid u : \mathbb{R}_0^+ \rightarrow U \}$, $f = \prod_{i=1}^{N} f_i, \sigma = \prod_{i=1}^{N} \sigma_i, r = \prod_{i=1}^{N} r_i, Y = \prod_{i=1}^{N} Y_i, h = \prod_{i=1}^{N} h_i$, and subject to the constraint:

$$w_{ij} = z_{ij}, \forall i \in \{1, \ldots, N\}, \forall j \in \{1, \ldots, N\} \setminus \{i\}. \quad (IV.3)$$

**Remark 4.2:** Although, without loss of generality, we assumed that only the external output values $y_i \in Y_i$ are accessible after the interconnection, one can assume that some of the internal output values $z_j \in Z_j$ are still accessible after the interconnection and, hence, they can also be considered as external output values as well (cf. the example in Section V).

Note that a network of stochastic hybrid systems can be viewed as an interconnection of stochastic hybrid subsystems as explained in the following remark.

**Remark 4.3:** Consider a stochastic hybrid system $\Sigma = (X, U, \mathcal{U}, f, \sigma, r, Y, h)$, where

$$f(x, u) = [f_1(x, u_1), \ldots, f_N(x, u_N)],$$

$$\sigma(x) = [\sigma_1(x_1), \ldots, \sigma_N(x_N)],$$

$$r(x) = [r_1(x_1); \ldots; r_N(x_N)],$$

$$h(x) = [h_1(x_1); \ldots; h_N(x_N)],$$

for any $x = [x_1; \ldots; x_N] \in X = \prod_{i=1}^{N} X_i$, and any $u = [u_1; \ldots; u_N] \in U = \prod_{i=1}^{N} U_i$, there exist $X_i, U_i$. Correspondingly, one can decompose $Y = \prod_{i=1}^{N} Y_i$, for some appropriate sets $Y_i$. One can readily verify that $\Sigma = \mathcal{I}(\Sigma_1, \ldots, \Sigma_N)$, where

$$\Sigma_i = \left( X_i, U_i \times \prod_{j \neq i=1}^{N} X_j, U \times D, f_i, \sigma_i, r_i, Y_i \times X_i, h_i \times 1_{X_i} \right),$$

and $D$ is a subset of the set of all measurable functions of time from intervals of the form $[0, \infty]$ to $\prod_{i=1}^{N} X_i$.

**B. Compositional approximations of interconnected systems**

This subsection contains the main contribution of the paper. We provide a bisimilar approximation of an interconnected system using bisimilar approximations of its subsystems. For showing the main result of the paper, we need the following assumption, inspired by the work in [14].

**Assumption 4.4:** For any $i, j \in \{1, \ldots, N\}$, there exist $K_\infty$ functions $\gamma_i$ and constants $\lambda_i \in \mathbb{R}^+$ and $\delta_{ij} \in \mathbb{R}_0^+$ such that one has:

$$\rho_i (2(N-1)\gamma_i^{-1}(s)) \leq \delta_{ij} \gamma_j(s), \quad \alpha_i(s) \geq \lambda_i \gamma_i(s),$$

for any $s \in \mathbb{R}_0^+$, where $\alpha_i, \rho_i$, and $\alpha_i$ are the functions appearing in Theorem 3.5, but with indices $i, j$ here.

To use in the next theorem, we define $\Lambda := \text{diag}(\lambda_1, \ldots, \lambda_N)$, $\Delta := \{ \delta_{ij} \},$ and $\Gamma(s) := [\gamma_1(s)_1; \ldots; \gamma_N(s)_N]$, where $s = [s_1; \ldots; s_N]$. We now have all the ingredients to show the main result of the paper.

**Theorem 4.5:** Consider $N$ stochastic hybrid subsystems $\Sigma_i$ and their corresponding approximations $\hat{\Sigma}_i, i \in \{1, \ldots, N\}$, whose input/output sets and output functions are decomposed as in (IV.1) and (IV.2). Assume $z_{ij}$ and $\gamma_{ij}$ are of equal dimension, $U_i = \overline{U}_i, \forall i \in \{1, \ldots, N\}, \forall j \in \{1, \ldots, N\} \setminus \{i\}$, and assume there exist stochastic bisimulation functions $\phi_i : X_i \times \overline{X_i} \rightarrow \mathbb{R}_0^+ \cup \{+\infty\}$ as in Theorem 3.5 between $\Sigma_i$ and $\hat{\Sigma}_i$ for any $i \in \{1, \ldots, N\}$. If Assumption 4.4 and $\hat{\rho}(\Delta^{-1} \Delta) < 1$ hold, then there exist constants $\mu_i \in \mathbb{R}^+$ such that $\Sigma_i = \{ \mu_i \phi_i \}$ is a stochastic bisimulation function between $\mathcal{I}(\Sigma_1, \ldots, \Sigma_N)$ and $\mathcal{I}(\hat{\Sigma}_1, \ldots, \hat{\Sigma}_N).

**Proof:** As showed in Lemma 3.1 in [14], if $\hat{\rho}(\Delta^{-1} \Delta) < 1$ holds, then there exist constants $\mu_i \in \mathbb{R}^+$, $i \in \{1, \ldots, N\}$, such that $\mu^T (-\Lambda + \Delta) < 0_N$, where $\mu := [\mu_1; \ldots; \mu_N]$. Let us first show that condition (i) in Definition 3.1 holds. For any $x := [x_1; \ldots; x_N] \in X_i$, $\tau := [\tau_1; \ldots; \tau_N] \in \overline{X}$, one obtains:

$$\| h(x) - \overline{h}(\tau) \| \leq \sum_{i=1}^{N} \| h_i(x_i) - \overline{h}_i(\tau_i) \|,$$

$$\leq \sum_{i=1}^{N} \| h_i(x_i) - \overline{h}_i(\tau_i) \| \leq \sum_{i=1}^{N} \alpha_i^{-1}(\phi_i(x_i, \tau_i)),$$

$$\leq \sum_{i=1}^{N} \alpha_i^{-1} \left( \frac{\mu_i \phi_i(x_i, \tau_i)}{\overline{\mu}} \right) \leq \sum_{i=1}^{N} \overline{\alpha}_i \left( \frac{\mu_i \phi_i(x_i, \tau_i)}{\overline{\mu}} \right),$$

where $\overline{\alpha}(s) = \mu \overline{\alpha}^{-1}(s/N), \forall s \in \mathbb{R}_0^+$, is a $K_\infty$ function, and condition (i) in Definition 3.1 is satisfied. Now we show that condition (ii) in Theorem 3.5 is satisfied. Consider any $x := [x_1; \ldots; x_N] \in X_i, \tau := [\tau_1; \ldots; \tau_N] \in \mathbb{R}_0^+$, and condition (i) in Definition 3.4 is satisfied. In order to show the chain of inequalities in (IV.4), we leverage the following inequality:

$$\rho_i(s_1 + \cdots + s_N) \leq \sum_{j=1}^{N} \rho_i(s_j + \cdots + s_j),$$

which is valid for any $\rho_i \in K_\infty$ and any $s_1, \ldots, s_N \in \mathbb{R}_0^+$. By defining

$$\alpha(s) := \min \left\{ -\mu^T (-\Lambda + \Delta) \Gamma(\phi_{\infty}(x, \tau)) \mid \mu^T \phi_{\infty}(x, \tau) = s \right\},$$

$$\rho(s) := \max \left\{ \sum_{i=1}^{N} \mu_{ij} \rho_i(2s_i) \mid \| [s_1; \ldots; s_N] \| = s \right\},$$

where $\phi_{\infty}(x, \tau) = [\phi_1(x_1, \tau_1); \ldots; \phi_N(x_N, \tau_N)]$, one obtain:

$$\mathcal{L}^n \phi_{\infty}(x, \tau) \leq -\alpha(\phi(x, \tau)) + \rho \| u - \pi \|,$$

which completes the proof. It is clear that $\alpha, \rho \in K_\infty$.

**Remark 4.6:** Note that if $\rho$ is a linear $K_\infty$ function, e.g. $\alpha(s) = s$ for any $s \in \mathbb{R}_0^+$, one can remove all the coefficients 2 and $N - 1$ in the proof of Theorem 4.5 as well as in Assumption 4.4.

**Remark 4.7:** Note that if $\Delta$ is irreducible [15], $\mu > 0$ can be chosen as a left eigenvector of $-\Lambda + \Delta$ corresponding to the largest eigenvalue, which is real and negative by the Perron-Frobenius theorem [15].

Figure 3 shows schematically the results of Theorem 4.5.
\[ L^u \Phi (x, x) = \sum_{i=1}^{N} \mu_i L^u \Phi_i (x_i, x_i) \leq \sum_{i=1}^{N} \mu_i (\alpha_i (\phi_i (x_i, x_i)) + \rho_i (\| w_i - m_i \| + \| u_i - m_i \|))) \]

\[ \leq \sum_{i=1}^{N} \mu_i (\alpha_i (\phi_i (x_i, x_i)) + \rho_i (2 \| w_i - m_i \| + 2 \| u_i - m_i \|)) \]

\[ \leq \sum_{i=1}^{N} \mu_i (\alpha_i (\phi_i (x_i, x_i)) + \rho_i (2 \| w_i - m_i \| + \| u_i - m_i \|)) \]

where the inequality follows from Lemma 2.3 in [16]. Hence, condition (i) in Definition 3.1 is satisfied with

V. EXAMPLE

Consider two stochastic hybrid systems \( \Sigma_1 \) and \( \Sigma_2 \) which are JLSS and whose dynamics are described as the following:

\[ \Sigma_i : \begin{cases} d \xi_i &= (A_i \xi_i + B_i u_i) dt + F_i \xi_i dW_i + R_i \xi_i dP_i, \\
\eta_i &= C_i \xi_i, \end{cases} \]

where

\[ A_i = 3.5 \begin{bmatrix} -2 & 1 & 1 \\ 1 & -1 & -2 \end{bmatrix}, \quad B_i = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad C_i^T = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \]

and \( F_i = R_i = 0.5I_i \) for \( i = 1, 2 \). The rate of the Poisson process \( P_i \) is \( \lambda = 0.5 \). For both systems \( \Sigma_1 \), we assume \( u_i = w_i, y_1 = y_2 = z_1 \) (cf. Remark 4.2), and \( y_2 = z_2 \). Let us show that \( \phi_i (\xi_i, \overline{\xi}_i) = \sqrt{\xi_i^T M \xi_i} \) is a stochastic bisimulation function as in Theorem 3.5 between \( \Sigma_i \) and \( \overline{\Sigma}_i \), where

\[ M = \begin{bmatrix} 1 & -1 & -1 & 0 \\ -1 & 2 & 2 & -1 \\ -1 & 2 & 2 & -1 \\ 0 & -1 & -1 & 1 \end{bmatrix}. \]

First let us remark that

\[ \phi_i (x_i, \overline{x}_i) = \sqrt{\xi_i^T M \xi_i} = \sqrt{\left( x_{i1} - x_{i2} + x_{i3} \right)^2 + \left( x_{i2} + x_{i3} - \overline{x}_{i1} \right)^2} \]

and

\[ \| x_{i1} - \overline{x}_{i1} \| \geq \sqrt{2} \| x_{i1} - x_{i2} \| = \sqrt{2} \| y_i - \overline{y}_i \|, \]

where the inequality follows from Lemma 2.3 in [16]. Hence, condition (i) in Definition 3.1 is satisfied with
Fig. 4. A few realizations of the error between the output of $\Sigma$ and of $\Sigma_i$, e.g. $|\eta_1 - \eta_1|$. The dashed line denotes the 93% confidence bound.

$$
\alpha_i(s) = \frac{\sqrt{\gamma}}{2} s \quad \text{for any } s \in \mathbb{R}_0^+.
$$
Moreover, one obtains

\[
L_{\mu_i} \phi_i(x_i, \pi_i) \leq \frac{1}{2} x^T M A x + x^T F^T M F x + \frac{1}{2} \phi(x_i, \pi_i) + \frac{1}{\lambda_i} \sqrt{x^T (R^2 M R + R^2 M + M R) x}
\]

where $x = [x_i; \pi_i]$, $u = [u_i; \bar{u}_i]$, $A = \text{diag}(A_i; \{ -3.5\})$, $B = \text{diag}(B_i; \{ 1\})$, $F = \text{diag}(F_i; \{ 0.5\})$, and $R = \text{diag}(R_i; \{ 0.5\})$. Therefore, condition (ii) in Theorem 3.5 is satisfied with $\alpha_i(s) = 2.86s$ and $\rho_i(s) = s$ for any $s \in \mathbb{R}_0^+$. Since $\rho_i$ is a linear $\mathcal{K}_{\infty}$ function (cf. Remark 4.6), by choosing $\delta_{ij} = \sqrt{2}$, $\gamma_i(s) = s$ for any $s \in \mathbb{R}_0^+$, and $\lambda_i = 2.86$ for $i, j = 1, 2$, one can readily verify that Assumption 4.4 holds. Since $\hat{\rho}(\Lambda^{-1}\Delta) = 0.99 < 1$ and using the results of Theorem 4.5, one concludes that there exists a vector $\mu = [\mu_1; \mu_2] > 0_2$ such that $\phi(x, \pi) = \mu_1 \phi_1(x_1, \pi_1) + \mu_2 \phi_2(x_2, \pi_2)$ is a stochastic bisimulation function between $\Sigma = \mathcal{I}(\Sigma_1, \Sigma_2)$ and $\tilde{\Sigma} = \mathcal{I}(\tilde{\Sigma}_1, \tilde{\Sigma}_2)$, where $x = [x_1; x_2]$ and $\pi = [\pi_1; \pi_2]$. Using Remark 4.7, one obtains $\mu_{i, 2} = \sqrt{2}$. In the simulations, the initial states of the interconnected systems $\Sigma$ and $\Sigma_0$ are chosen as $x_0 = \{ -0.87; -2.71; -3.28; -0.22; -0.59; -0.71\}$ and $\pi_0 = \{ 1.83; 0.32\}$. In figure 4, we show a few realizations of the error between the output of $\Sigma$ and of $\Sigma_i$, e.g. $|\eta_1 - \eta_1|$. The dashed line denotes the 93% confidence bound given by the computed stochastic bisimulation function $\phi$ as in inequality (III.1).

VI. DISCUSSION

In this paper, we provided a compositional approach for approximations of interconnected stochastic hybrid systems. Given stochastic hybrid subsystems and their corresponding approximations and the quantified errors between them, we provided the approximations of an interconnection of those stochastic hybrid subsystems and the overall approximation error. Therefore, one can leverage the proposed techniques in this paper to potentially solve safety/reachability problems for large-scale stochastic hybrid systems which are interconnection of several stochastic hybrid subsystems. Note that the considered approximate abstractions in this paper are still infinite but possibly simpler (lower state space dimension). The author is currently investigating the extensions of the proposed work here to provide finite approximations of interconnected stochastic hybrid systems by using the finite approximations of each stochastic hybrid subsystems proposed in [17], [18].

VII. ACKNOWLEDGEMENT

The author would like to thank Peyman Mohajerin Esfahani for fruitful technical discussions.

REFERENCES


