Results on robust stability and feedback stabilization for systems with a continuum of equilibria

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Abstract—A discrete-time dynamical system, with a continuum of equilibria and nonlinear, multivalued dynamics is discussed. An asymptotic stability property, its robustness, and necessary and sufficient Lyapunov-like conditions are presented. The conditions involve a set-valued Lyapunov function. Then a control system is studied, in which the stability property can be achieved by open-loop controls, and a feedback control construction is presented. Set-valued control Lyapunov functions are introduced, for the purpose of robust feedback design.

I. INTRODUCTION

Given a dynamical system, a Lyapunov stable equilibrium surrounded by points from which every solution converges to a possibly different equilibrium, was called semistable in [4]. Semistability is a proper stability concept for systems which exhibit a continuum of equilibria, and was further studied in [15], [16], [17], [5], and more, where sufficient conditions, some necessary but not sufficient conditions, and finite-time semistability have been obtained.

First necessary and sufficient conditions for semistability, renamed as pointwise asymptotic stability, were given in [12] in terms of a set-valued Lyapunov function, in the setting of discrete-time multivalued dynamics. The set-valued approach was motivated by [22], [3], etc., where the decrease of the convex hull of positions of agents, or other related sets, was proposed as a sufficient condition for consensus.

Robustness of pointwise asymptotic stability has seen limited treatment. The converse result of [15] was used in [18] to state robustness to higher-order perturbations under homogeneity assumptions. This result assumed Lyapunov stability of the equilibria for the perturbed dynamics. A related result was given in [14] for a switching system. For the classical asymptotic stability, robustness was tied by [10] to the existence of regular Lyapunov functions, in the setting of differential inclusions, where regularity meant smoothness. This idea carried over to the setting of difference inclusions [20] and hybrid dynamics [6]. For pointwise asymptotic stability, this idea was used by [13], where the necessary and sufficient conditions [12] led to an equivalent characterization in terms of a continuous set-valued Lyapunov function. As a consequence, one obtains that pointwise asymptotic stability of a compact set is robust for continuous, but set-valued, difference inclusions. Results of [12], [13] are recalled here in Section II.

II. POINTWISE ASYMPTOTIC STABILITY

In this section, the difference inclusion

\[ x^+ \in F(x), \tag{1} \]

is considered, where \( F : \mathbb{R}^n \to 2^{\mathbb{R}^n} \) is a set-valued mapping, i.e., for every \( x \in \mathbb{R}^n \), \( F(x) \) is a subset of \( \mathbb{R}^n \). It is assumed that for every \( x \in \mathbb{R}^n \), \( F(x) \neq \emptyset \). The function \( \phi : N_0 \to \mathbb{R}^n \), where \( N_0 = N \cup \{0\} \), is a solution to (1) from the initial point \( \xi \in \mathbb{R}^n \) if \( \phi(0) = \xi \) and, for all \( i \in \mathbb{N} \), \( \phi(i) \in F(\phi(i - 1)) \). Throughout the paper, \( A \subset \mathbb{R}^n \) is a nonempty set.

Definition 2.1: The set \( A \) is pointwise asymptotically stable (PAS) for (1) if

- every \( a \in A \) is Lyapunov stable for (1), that is, for every \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that every solution \( \phi \) with \( ||\phi(0) - a|| < \delta \) satisfies \( ||\phi(j) - a|| < \varepsilon \) for every \( j \in N_0 \), and
- every solution \( \phi \) to (1) is convergent and \( \lim_{j \to \infty} \phi(j) \in A \).

Example 2.2: In \( \mathbb{R}^2 \), let

\[ A = \{x \mid x_1 = x_2\}, \]
which is the usually considered set of consensus states. This set is pointwise asymptotically stable for the dynamics
\[ x_1^+ = \frac{3}{4} x_1 + \frac{1}{4} x_2, \quad x_2^+ = \frac{3}{4} x_1 + \frac{1}{4} x_2 \]
which represent the \( i \)-th agent, with \( i = 1, 2 \), moving with each step halfway to the average state
\[ x_{\text{ave}} = \frac{1}{2} (x_1 + x_2). \]
Hence, \( x^+ = x + \frac{1}{2} (a(x) - x) = \frac{1}{2} (a(x) + x) \), where
\[ a(x) = (x_{\text{ave}}, x_{\text{ave}}). \]
Note that \( x_{\text{ave}} = x_{\text{ave}}^+ \), where the latter denotes the average of \( x_1^+ \) and \( x_2^+ \), and so \( a(x) = a(x^+) \).

Example 2.3: In \( \mathbb{R}^2 \), let
\[ A = \{ x \mid -1 \leq x_1 = x_2 \leq 1 \}, \]
which is a compact subset of the consensus set considered in Example 2.2. Let \( P_A(x) \) be the point in \( A \) closest to \( x \) in the Euclidean norm:
\[ P_A(x) = \begin{cases} (-1, -1) & \text{if } x_1 + x_2 \leq -2 \\ a(x) & \text{if } -2 \leq x_1 + x_2 \leq 2 \\ (1, 1) & \text{if } 2 \leq x_1 + x_2 \end{cases} \]
Then \( A \) is pointwise asymptotically stable for the dynamics
\[ x^+ = x + \frac{1}{2} (P_A(x) - x) = \frac{1}{2} (P_A(x) + x), \]
which represent the movement of \( x \) halfway toward \( P_A(x) \) and are nonlinear. Note that \( P_A(x) = P_A(x^+) \).

Example 2.4: The set \( A \) from Example 2.3 is pointwise asymptotically stable for the dynamics where \( x^+ \) is given by
\[ \alpha(x)(a(x) + (1 - \alpha(x))x) \text{ if } |x_1 + x_2| > 2, x \neq a(x) \]
\[ \frac{1}{2} (P_A(x) + x) \text{ otherwise}, \]
where, for \( x \neq a(x) \),
\[ \alpha(x) = \max \left\{ \frac{\|x - P_A(x)\|}{2\|x - a(x)\|}, 1 \right\}. \]
Thus, if \( -2 \leq x_1 + x_2 \leq 2 \), then \( x^+ = \frac{1}{2} (a(x) + x) \), and the dynamics agree with Example 2.2 and because then \( a(x) = P_A(x) \), with Example 2.3. If \( |x_1 + x_2| > 2 \) and \( x = a(x) \), with the latter condition equivalent to \( x_1 = x_2 \), \( x^+ = \frac{1}{2} (P_A(x) + x) \) and the dynamics agree with Example 2.3. For example, when \( x_1 + x_2 < -2 \) and \( x_1 = x_2 \), then dynamics represent moving halfway toward \((-1, -1)\). If \( |x_1 + x_2| > 2 \) and \( x \neq a(x) \) (i.e., \( x_1 \neq x_2 \)), the dynamics represent moving to \( a(x) \) if the distance from \( x \) to \( a(x) \) is not greater than half the distance from \( x \) to \( P_A(x) \), and moving in the direction of \( a(x) \) with the size of the step \( \|x^+ - x\| \) equal to half the distance from \( x \) to \( P_A(x) \) otherwise. This gives finite-time convergence to the set \( x_1 = x_2 \) from points with \( |x_1 + x_2| > 2 \) and then exponential convergence to either \((-1, -1)\) or \((1, 1)\). From points with \( |x_1 + x_2| \leq 2 \), exponential convergence to \( a(x) \) occurs.

A. Set-valued Lyapunov functions

The following definition comes from [12, Definition 2.3], with a minor change allowing for lack of regularity of \( W \) at points not in \( A \).

Definition 2.5: A set-valued mapping \( W : \mathbb{R}^n \rightrightarrows \mathbb{R}^n \) is a set-valued Lyapunov function for (1) and the set \( A \) if
(a) \( W(x) = \{ x \} \) for every \( x \in A \);
(b) \( x \in W(x) \) for every \( x \in \mathbb{R}^n \);
(c) \( W \) is locally bounded and, at every \( x \in A \), is outer semicontinuous;
(d) there exists a continuous and positive definite with respect to \( A \) function \( \alpha : \text{dom} W \to [0, \infty) \) such that, for any convergent sequence of points \( x_i \), satisfying \( \lim_{i \to \infty} \alpha(x_i) = 0 \), one has \( \lim_{i \to \infty} x_i \in A \) and
\[ W(F(x)) + \alpha(x) \mathbb{B} \subseteq W(x) \quad \forall x \in \mathbb{R}^n. \]

Above, \( \mathbb{B} \) is the unit ball in \( \mathbb{R}^n \) centered at 0, and \( W(F(x)) = \bigcup_{x \in F(x)} W(x) \). Outer semicontinuity of \( W \) at \( x \) means that for every sequence \( x^i \to x \) and every convergent sequence \( y^i \in W(x^i) \), one has \( \lim y^i \in W(x) \). Local boundedness of \( W \) means that for every \( x \in \mathbb{R}^n \) there exists a neighborhood \( U \) of \( x \) such that \( W(U) \) is a bounded set. See [25] for more details. An outer semicontinuous (continuous) set-valued Lyapunov function \( W \) is a set-valued Lyapunov function \( W \) which is outer semicontinuous (continuous) at every \( x \in \text{dom} W \). Note that the existence of the continuous and positive definite with respect to \( A \) function \( \alpha \) implies that \( A \) must be closed.

Example 2.6: With \( A \) as in Examples 2.3, 2.4, consider the set-valued mapping \( W : \mathbb{R}^n \rightrightarrows \mathbb{R}^n \) given by
\[ W(x) = P_A(x) + \|x - P_A(x)\| \mathbb{B}. \]
In other words, \( W(x) \) is a closed ball centered at \( P_A(x) \), the point in \( A \) closest to \( x \), with the radius equal to the distance from \( x \) to \( A \). Clearly, \( x \in W(x) \) for every \( x \in \mathbb{R}^n \) and \( \{x\} = W(x) \) if and only if \( x \in A \). It is straightforward that \( W \) is locally bounded. Because \( A \) is a convex set, \( P_A \) is continuous, and hence \( W \) is not just outer semicontinuous but in fact continuous, as a set-valued mapping, on \( \mathbb{R}^n \).

For the dynamics in Example 2.3, one has
\[ W(x^+) = P_A(x^+) + \|x^+ - P_A(x^+)\| \mathbb{B} = P_A(x) + \frac{1}{2} \|x - P_A(x)\| \mathbb{B} \]
and hence
\[ W(x^+) + \frac{1}{2} \|x - P_A(x)\| \mathbb{B} \subseteq W(x). \]
Thus \( W \) is a set-valued Lyapunov function, with (d) of Definition 2.5 satisfied with \( \alpha(x) = \frac{1}{2} \|x - P_A(x)\| \).

For the dynamics in Example 2.4, \( W \) is not a set-valued Lyapunov function. Indeed, take \( x \notin A \) with \( x_1 = x_2 \) and a sequence \( x^i \to x \) with \( a(x^i) = x \) and \( x^i_1 \neq x^i_2 \). For large \( i \) one has \( P_A(x^i) = P_A(x), x^i = x \), and \( \|x^i - P_A(x^i)\| \to \|x - P_A(x)\| \). Continuity of \( \alpha \), as requested in (d) of Definition 2.5, implies then that \( \alpha(x) = 0 \), which violates the condition of positive definiteness of \( \alpha \). \( \triangle \)
Example 2.7: Consider $A$ as in Example 2.4, and recall that $W$ from Example 2.6 is not a set-valued Lyapunov function for the dynamics in Example 2.4. Consider $W_1 : \mathbb{R}^n \to \mathbb{R}^n$, where $W_1(x)$ is given by
\[
W_1(x) = \begin{cases} 
W(x) + \|x - P_A(x)||\mathbb{B} & \text{if } |x_1 + x_2| > 2, x \neq a(x) \\
W(x) & \text{otherwise}
\end{cases}
\]
In other words, in the first case above, $W_1(x) = P_A(x) + 2\|x - P_A(x)||\mathbb{B}$. The function $W_1$ is locally bounded, not continuous, in fact not outer semicontinuous on $\mathbb{R}^n$, but it is outer semicontinuous at every point $x \in A$. The function $W_1$ turns out to be a set-valued Lyapunov function for the dynamics in Example 2.4, with $\alpha(x) = \frac{1}{4}\|x - P_A(x)||$. The decrease condition is satisfied at points $x$ where $|x_1 + x_2| \leq 2$ or $x = a(x)$, because at such $x$, the dynamics agree with those in Example 2.3, and for these dynamics, the decrease was checked in Example 2.6, with a larger $\alpha$. For points where $|x_1 + x_2| > 2, x \neq a(x)$, and $x^+ = a(x^+), W_1(x^+) = P_A(x) + \|x^+ - P_A(x)||\mathbb{B}, W_1(x) = P_A(x) + 2\|x - P_A(x)||\mathbb{B}$, and because $\|x^+ - P_A(x)|| < \|x - P_A(x)||$, $W_1(x^+) + \|x - P_A(x)||\mathbb{B} \subset W_1(x)$
which implies the decrease condition. When $|x_1 + x_2| > 2, x \neq a(x)$, and $x^+ \neq a(x^+)$, some geometry and calculus shows that $\|x^+ - P_A(x)|| \leq \frac{1}{4}\|x - P_A(x)||$, and then $2\|x^+ - P_A(x)|| + \frac{1}{4}\|x - P_A(x)|| \leq 2\|x - P_A(x)||$, which implies $W_1(x^+) + \frac{1}{4}\|x - P_A(x)||\mathbb{B} \subset W_1(x)$. △

The statements below, giving necessary and sufficient conditions for pointwise asymptotic stability in terms of set-valued Lyapunov functions comes from [12] and [13].

Theorem 2.8:
(a) If there exists a set-valued Lyapunov function $W$ for (1) and $A$, then $A$ is pointwise asymptotically stable.
(b) Assume that $F$ is locally bounded and outer semicontinuous, and $A$ is compact. If $A$ is pointwise asymptotically stable then there exists an outer semicontinuous set-valued Lyapunov function $W$ for (1) and $A$.
(c) Assume that $F$ is locally bounded and continuous, and $A$ is closed. If $A$ is pointwise asymptotically stable then there exists a continuous set-valued Lyapunov function $W$ for (1) and $A$.

Continuity of the set-valued mapping $F$ assumed in (c) above means that $F$ is outer semicontinuous, as discussed below Definition 2.5, and also inner semicontinuous: for every $x \in \mathbb{R}^n$, every $y \in F(x)$, and every sequence $x^i \to x$, there exist $y^i \in F(x^i)$ such that $y^i \to y$.

B. Robustness

The definition of robustness given below is parallel to what has been considered for differential inclusions [10], difference inclusions [20], and hybrid dynamics [6], when discussing asymptotic stability. Essentially, robustness of pointwise asymptotic stability means that the asymptotic stability also holds when the dynamics are “enlarged” to account for small perturbations.

Definition 2.9: The set $A$ is robustly pointwise asymptotically stable for (1) if it is pointwise asymptotically stable and there exists a function $\rho : \mathbb{R}^n \to [0, \infty)$ which is continuous and positive definite with respect to $A$ and such that $A$ is pointwise asymptotically stable for
\[
x^+ \in F_\rho(x),
\]
where $F_\rho : \mathbb{R} \to \mathbb{R}^n$ is a set-valued mapping defined, at each $x \in \mathbb{R}^n$, by
\[
F_\rho(x) = \bigcup_{y \in F(x + \rho(x))} y + \rho(y)\mathbb{B}.
\]
It turns out that, in a manner similar to what is known for asymptotic stability for continuous-time, discrete-time, or hybrid dynamics and regular Lyapunov functions, pointwise asymptotic stability is robust whenever this property can be characterized by a continuous set-valued Lyapunov function. The result below and the corollary come from [13].

Theorem 2.10: Let $F$ be locally bounded and $A$ be compact. The following are equivalent:
(a) $A$ robustly pointwise asymptotically stable for (1).
(b) There exists a continuous set-valued Lyapunov function $W$ for (1) and $A$.

Combining this with Theorem 2.8 gives the following.

Corollary 2.11: Suppose that $F$ is locally bounded and continuous and $A$ is compact. If $A$ is pointwise asymptotically stable for (1) then $A$ is robustly pointwise asymptotically stable for (1).

Example 2.12: Pointwise asymptotic stability in Example 2.3 is robust, because the set-valued Lyapunov function exhibited in Example 2.6 is continuous. The robustness could be argued directly, relying on the continuity of the dynamics, but to the best of the authors knowledge, there are no general results addressing this.

Pointwise asymptotic stability in Example 2.4 is not robust. Indeed, for any function $\rho$ as in Definition 2.9, for any $x$ with $x_1 = x_2, x \notin A$, the dynamics (3) generate solutions changing around $x$. Accordingly, there exist no continuous set-valued Lyapunov function $W$ in this case, and consequently, Example 2.7 suggested a $W$ that is not continuous. The lack of robustness can be also seen by considering the set-valued regularization of the discontinuous dynamics in Example 2.4, following the ideas of Filippov [11] and Krasovskii [21]. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be the discontinuous function representing the dynamics in Example 2.4. Consider the set-valued mapping $F : \mathbb{R}^2 \to \mathbb{R}^2$ given by
\[
F(x) = \bigcap_{\delta > 0} f(x + \delta\mathbb{B}).
\]
Then, for every $x$ with $x_1 = x_2, x \notin A$, one has $x \in F(x)$, and hence each such point is an equilibrium for $x^+ \in F(x)$. Hence, for this inclusion, the set $A$ of Example 2.4 is not pointwise asymptotically stable, and this reflects the lack of robustness for the original, discontinuous dynamics. △
III. Feedback Pointwise Stabilization

For the discrete-time nonlinear control system
\[ x^{+} = f(x, u), \ u \in U, \]  
(5)
where \( f : \mathbb{R}^n \times \mathbb{R}^k \) is a function and \( U \subset \mathbb{R}^k \) is a set, this section addresses the problem of construction of feedback which renders a set \( A \) pointwise asymptotically stable.

**Assumption 3.1:** The function \( f : \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^n \) is continuous. The set \( U \subset \mathbb{R}^k \) is nonempty and compact.

The assumption below requires a kind of asymptotic controllability, similar to what is assumed for nonlinear control systems and the question of feedback stabilization and asymptotic stability, and appropriate for the case of pointwise asymptotic stability.

**Assumption 3.2:** The set \( A \) is compact and:

(a) for every \( \xi \in \mathbb{R}^n \), there exists an open-loop control \( u_\xi : \mathbb{R}_0 \to U \) such that the resulting solution \( \phi_\xi : \mathbb{R}_0 \to \mathbb{R}^n \) to (5) converges, and
\[ \lim_{j \to \infty} \phi_\xi(j) \in A; \]
(b) for every \( a \in A \), for every \( \varepsilon > 0 \), there exists \( \delta > 0 \) such that, for every \( \xi \) with \( \|\xi - a\| < \delta \), the solution \( \phi_\xi \) from (a) is such that \( \|\phi_\xi(j) - a\| < \varepsilon \) for every \( j \in \mathbb{N}_0 \).

Some technical consequences of these assumptions are required for the construction of the mentioned feedback.

**Lemma 3.3:** Under Assumption 3.2, for every \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that, for every \( a \in A \), if \( \|\xi - a\| < \delta \) then the solution \( \phi_\xi \) from Assumption 3.2 satisfies \( \|\phi_\xi(j) - a\| < \varepsilon \) and \( \|\phi_\xi(j) - \phi_\xi(j')\| < \varepsilon \) for every \( j, j' \in \mathbb{N}_0 \).

**Lemma 3.4:** Under Assumption 3.1 and Assumption 3.2, for every \( a \in A \) there exists \( u_a \in U \) such that \( f(a, u_a) = a \).

One can argue that under Assumptions 3.1, 3.2 there exists a feedback \( u_0 : \mathbb{R}^n \to U \) such that, for the closed-loop system \( x^{+} = f(x, u(x)) \), the set \( A \) is asymptotically stable. A weaker result is sufficient for the developments here. It ensures that a feedback can cause solutions to enter a neighborhood of \( A \).

**Lemma 3.5:** Under Assumption 3.2 (a), for every \( \delta > 0 \), there exists a feedback mapping \( u_0 : \mathbb{R}^n \setminus (A + \delta \mathbb{B}) \to U \) such that every solution \( \phi \) to the closed-loop system \( x^{+} = f(x, u(x)) \) that \( \phi(0) \notin A + \delta \mathbb{B} \) satisfies, for some \( j \in \mathbb{N}_0 \), \( \|\phi(j)\| < \delta \).

The main result of the section follows. The construction, given in detail in the proof, is similar to the patchy feedback construction of [1]. The verification that the feedback accomplishes pointwise asymptotic stability is omitted.

**Theorem 3.6:** Under Assumption 3.1 and Assumption 3.2 there exists a feedback mapping \( u : \mathbb{R}^n \to U \) such that, for the closed-loop system
\[ x^{+} = f(x, u(x)), \]
(6)
the set \( A \) is pointwise asymptotically stable.

**Proof:** For each \( \xi \in \mathbb{R}^n \), let \( u_\xi \) and \( \phi_\xi \) satisfy the conditions of Assumption (3.2). For \( k \in \mathbb{N}_0 \), invoke Lemma 3.3 to obtain a strictly decreasing sequence of \( \delta_k > 0 \) such that, for every \( a \in A \), if \( \|\xi - a\| < \delta_k \) then \( \phi_\xi \) satisfies
\[ \|\phi_\xi(j) - a\| < 2^{-k} \text{ and } \|\phi_\xi(j) - \phi_\xi(j')\| < 2^{-k} \]
for every \( j, j' \in \mathbb{N}_0 \). For \( k = 1, 2, \ldots \), let
\[ R_k = \{ x \in \mathbb{R}^n | \delta_k+1 \leq \|x\| A \leq \delta_k \}. \]
Fix \( k = 1, 2, \ldots \), consider the following construction. For every \( \xi \in R_k \), let \( J_\xi \in \mathbb{N}_0 \) denote the least \( j \) such that \( \|\phi_\xi(j)\| A < \delta_{k+1} \). Such \( J_\xi \) exists because the limit of \( \phi_\xi \) belongs to \( A \). Recall that \( \|\phi_\xi(j)\| A < 2^{-k} \) for every \( j \in \mathbb{N}_0 \) and let open neighborhoods \( O_{\xi, m}, m = 0, 1, \ldots, J_\xi, \) of points \( \phi_\xi(m) \) be such that
1. \( f(O_{\xi, m}, u_\xi(m)) \subset O_{\xi, m+1} \) for every \( m = 0, 1, \ldots, J_\xi - 1, \)
2. \( \delta_{k+2} = \|O_{\xi, m}\| A < 2^{-k} \) for every \( m = 0, 1, \ldots, J_\xi - 1, \)
3. \( \|O_{\xi, J_\xi}\| A < \delta_{k+1}, \)
4. for every \( m = 0, 1, \ldots, J_\xi - 1 \) and every \( \eta \in O_{\xi, m}, \)
   the solution \( \psi : \{m, m+1, \ldots, J_\xi \} \to \mathbb{R}^n \) to \( \psi^+ = f(\psi, u_\xi(j)), j = m, m+1, \ldots, J_\xi - 1 \) satisfies \( \|\psi(j) - \psi(j')\| A < 2^{-k} \) for every \( j, j' \in \{m, m+1, \ldots, J_\xi \}. \)

This is possible because \( f \) is continuous. Let \( O_{\xi, k} = \bigcup_{m=0}^{k} O_{\xi, m} \) and pick finitely many \( \xi_l \in R_k, l = 1, 2, \ldots, L_k, \) so that the open sets \( O_{\xi_l} \) form an open cover of \( R_k \). This is possible because \( R_k \) is compact. For \( l = 1, 2, \ldots, L_k, m = 0, 1, \ldots, M_{k,l} := J_\xi_l - 1, \) let \( O_{k,l,m} \) be the open neighborhood \( O_{\xi_l, m} \) and let \( u_{k,l,m} = u_{\xi_l, m} \).

Repeating this construction for every \( k = 1, 2, \ldots \), yields, for \( k = 1, 2, \ldots, l = 1, 2, \ldots, L_k, \) and \( m = 0, 1, \ldots, M_{k,l} \) open sets \( O_{k,l,m} \) and control values \( u_{k,l,m} \). Note that, by construction \( \delta_{k+2} < \|O_{k,l,m}\| A < 2^{-k} \) and so \( O_{k,l,m} \cap A = \emptyset \) and the family of sets \( O_{k,l,m} \) is locally finite on \( \mathbb{R}^n \). For every \( \xi \in \bigcup_{k,l,m} O_{k,l,m}, \) let \( k_\xi, l_\xi, m_\xi \) be the lexicographically greatest index of all sets \( O_{k,l,m} \) containing \( \xi \). That is, \( \xi \in O_{k_\xi, l_\xi, m_\xi} \) and if \( \xi \in O_{k, l, m} \) for a different index, then \( k_\xi > k \) or \( k_\xi = k \) and \( l_\xi > l \) or \( l_\xi = k, l_\xi = l, \) and \( m_\xi > m \). Let \( u_\xi : \mathbb{R}^n \to U \) be a feedback coming from Lemma 3.5 corresponding to \( \delta_1 \) and \( u_\xi \) be such that \( f(x, u_\xi) = x \) for \( x \in A, \) following Lemma 3.4. Finally, define the function \( u : \mathbb{R}^n \to U \) by
\[ u(x) = \begin{cases} u_{k_\xi, l_\xi, m_\xi} & \text{if } x \in \bigcup_{k,l,m} O_{k,l,m} \\ u_0(x) & \text{if } x \notin A \cup \bigcup_{k,l,m} O_{k,l,m} \\ u_x & \text{if } x \in A \end{cases} \]
The discussion above ensures that \( u \) is well-defined, and the right-hand side of (6) is nonempty for every \( x \in \mathbb{R}^n \). The remaining arguments, that the feedback yields pointwise asymptotic stability, are omitted.

IV. Set-Valued Control Lyapunov Functions

This section begins with a definition of a set-valued control Lyapunov function for the control system (5), building on the definition of a set-valued Lyapunov function for the difference inclusion (1).
Definition 4.1: A set-valued mapping $W : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is a set-valued control Lyapunov function for (5) and the set $A$ if it satisfies (a), (b), and (c) of Definition 2.5 and (d') there exists a continuous and positive definite with respect to $A$ function $\alpha : \text{dom} W \to [0, \infty)$ such that, for any convergent sequence of points $x_i$ satisfying $\lim_{i \to \infty} \alpha(x_i) = 0$, one has $\lim_{i \to \infty} x_i \in A$ and such that, for every $x \in \mathbb{R}^n$ there exists $u \in U$ such that

$$W(f(x, u)) + \alpha(x)B \subset W(x).$$

Thus, $W$ is a set-valued control Lyapunov function for (5) if, for some function $u : \mathbb{R}^n \to U$, $W$ is a set-valued Lyapunov function for the closed-loop (6). Theorem 2.8 then implies that, for (6), $A$ is PAS. If $W$ is also continuous, Theorem 2.10 implies that the pointwise asymptotic stability is robust. These conclusions do not depend on continuity or semicontinuity of the closed-loop dynamics.

Theorem 4.2: If there exists a set-valued control Lyapunov function $W$ for (5) and $A$, then there exists a function $u : \mathbb{R}^n \to U$ such that $W$ is pointwise asymptotically stable for (6). Furthermore, if $W$ is continuous, $A$ is compact, and $x \mapsto f(x, u(x))$ is locally bounded, then $A$ is robustly pointwise asymptotically stable for (6).

Example 4.3: Consider the control system (5) with $f : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$ and $U \subset \mathbb{R}^2$ be given by

$$f(x, u) = x + \frac{1}{2}\|x - P_A(x)\|u, \quad U = B,$$

where $A$ is the set considered in Examples 2.3, 2.4. The function $W$ defined in Example 2.6 is a set-valued control Lyapunov function for this control system. Indeed, the dynamics in Example 2.6 correspond to the choice of

$$u(x) = \frac{P_A(x) - x}{\|P_A(x) - x\|}$$

if $x \notin A$, $u(x) = 0$ if $x \in A$, and for these dynamics, $W$ is a set-valued Lyapunov function. Note that the dynamics of Example 2.4 come from a different choice of $u$, and for that $u$, $W$ is not a set-valued Lyapunov function, as argued in Example 2.7.

V. CONCLUSIONS

The paper has presented material on set-valued Lyapunov functions for pointwise asymptotic stability of a closed set. New contributions are feedback pointwise stabilization of a set, for a general nonlinear control system and under an assumption of open-loop pointwise stabilization, and the concept of a set-valued control Lyapunov function. Future work is on robust stabilization, which may be possible through the ideas of [23], [24], where the patchy feedback of [1] was altered to ensure robustness, and to employ optimization and optimal control, see [20], [20] and the references therein, or other approaches, for example as in [7] for continuous-time, for the construction of such control Lyapunov functions.