The consensus problem in the behavioral approach

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Abstract— In this paper we present some preliminary results about the consensus problem in the behavioral approach. Specifically, we consider a group of \( N \) homogeneous agents whose dynamics are described by the same (linear and time-invariant) behavioral model, involving inputs, measurable variables and target variables. We assume that the agents’ mutual communication is described by some (not necessarily symmetric) adjacency matrix. In this set-up we have derived necessary and sufficient conditions for a group of homogenous dynamic controllers, making use of the weighted information received by each agent about the other agents’ dynamics, for the \( N \) agents to achieve consensus on the target variables dynamics (under regularity constraints on the overall interconnection). Such conditions have been extended to the case when consensus is searched for both the target and the measurable variables. Finally, it is shown that the paper results encompass, as a special case, the classical situation when the agents’ dynamics are described by state-space models and the target (measurable) variables are the state (output) variables.

I. INTRODUCTION

The mathematical formulation of multi-agents systems and consensus problems was introduced several years ago in some pioneering papers as [3], [13], [14]. But it was only a decade ago that a wide stream of research on these topics started, thanks to milestone contributions such as [4], [6], [7], [8], [9]. Aside from the theoretical challenges that these problems pose, strong motivations for such a widespread interest come from the numerous application problems that can be naturally stated as consensus problems. Indeed, when dealing with sensor networks, coordination of mobile robots or UAVs, flocking and swarming in animal groups, dynamics of opinion forming, etc. the main control target can always be mathematically formalized as a consensus problem among agents, exchanging information and resorting to distributed algorithms that make use of the information collected from neighboring agents (see, e.g. [11], [12]).

While the first contributions on this subject focused on agents described as simple or double integrators, more recent works addressed the case of agents described by higher order models [4], [12], [15], [16], [17]. The vast majority of the literature on consensus, however, assumes that the homogeneous agents dynamics is described by a state-space model and that consensus is achieved through a static state- or output-feedback, that makes use of the weighted information collected from the neighboring agents, in a cooperative set-up (see [1] for consensus under antagonistic interactions).

The aim of this paper is to investigate the multi-agent consensus problem in a broader context, by not only assuming for the agents an extremely general model, but also assuming that the distributed controllers are described by higher order dynamic models. In addition, we do not impose any constraint on the adjacency matrix describing the agents interactions, so that both cooperative and antagonist relationships are possible. Finally, we distinguish the variables that are the target of the consensus problem from the measured variables used to exchange information, but we address also the case when consensus has to be achieved on both of them.

In order to investigate this general problem, the behavioral approach developed by Jan Willems [10], [18], [19] seems to be the most appropriate set-up. Since this set-up has never been used before in this context, we have tried to make the paper as self-contained as possible, by recalling the few fundamental definitions and results that are necessary to understand the technical details of the paper. A comprehensive treatment of the behavior theory can be found in any of the three aforementioned references.

In detail, the paper is organized as follows. In section II preliminary definitions and notation are given. In section III the consensus problem is posed. Section IV provides a characterization of the controllers that allow to achieve consensus on the variable \( x \), by means of a regular interconnection, and section V provides the analogous characterization for controllers that achieve consensus also on the measured variable \( y \). Section VI concludes the paper by showing how some classical results on consensus easily follow from the present analysis. It is worthwhile to remark that the results provided in this paper do not only hold for standard state-space models, but for every linear time-invariant description, in particular for singular (descriptor) systems and for polynomial matrix descriptions, and it does not require any form of causality or properness.

II. PRELIMINARIES

We introduce some basic notation and the main definitions that will be used in the following. We refer to [10] for a comprehensive treatment of the basics of the behavioral approach introduced by Jan Willems [18].

Given a \((k \times q)\)-matrix \( R \), \( R_{i \cdot} \) denotes the \( i \)th row of \( R \) for \( i \in \{1, \ldots, k\} \), and \( R_{\cdot j} \) denotes the \( j \)th column for \( j \in \{1, \ldots, q\} \). For the \((k \times k)\)-identity matrix we write \( I_k \). The \( k \)-dimensional vector with all unitary entries is denoted by \( 1_k \), while the \( i \)th standard basis vector in \( \mathbb{C}^k \) is denoted by \( e_i \). The spectrum of a square matrix \( L \) will be denoted by \( \text{spec}(L) \). For any \( k \)-dimensional vector \( v = (v_1, v_2, \ldots, v_k)^\top \), \( \text{diag}(v) \) is the diagonal matrix with diagonal entries \( v_1, v_2, \ldots, v_k \).

We let \( \mathcal{D} \) denote the ring of polynomials in the indeterminate \( s \) with either real or complex coefficients (depending...
on the context), namely $I = \mathbb{R}[s]$ or $\mathbb{C}[s]$.

The symbol $I \subset \mathbb{C}$ will denote the stability region of the complex plane. Classical examples of stability regions $I$ are the following ones:

$I_{c.a.} := \{ \lambda \in \mathbb{C} : \text{Re}(\lambda) < -a \}$ for $a \in \mathbb{R}_+$,

$I_{s.a.} := \{ \lambda \in \mathbb{C} : \left| \text{Re}(\lambda) \right| < -a \}$ for $a \in \mathbb{R}_+$.

Given a polynomial $p \in I$, we say that $p$ is $I$-stable if all its zeros belong to $I$.

In the paper we will consider continuous-time signals defined on the time set $\mathbb{R}$. Signals will be either real or complex valued and hence they will be, in general, elements of $(\mathbb{R}^q)^R$ or $(\mathbb{C}^q)^R$, for some $q \in \mathbb{N}$. By $\mathcal{H}^q$ we denote the set of arbitrarily often differentiable functions, $C^{\infty}(\mathbb{R}, \mathbb{R}^q) \subseteq (\mathbb{R}^q)^R$ (or $C^{\infty}(\mathbb{R}, \mathbb{C}^q) \subseteq (\mathbb{C}^q)^R$).

For every $R = \sum_{i=0}^q R_i x^i \in \mathbb{R}^{k \times q}$, we associate with $R$ the polynomial differential operator $R(\frac{d}{dt}) = \sum_{i=0}^q R_i \frac{d^i}{dt^i}$. The action of such a polynomial matrix differential operator $R$ on any signal $w \in \mathcal{H}^q$ is denoted by $R \circ w$.

We will consider linear, time-invariant behaviors [10] described as the kernels of polynomial matrix operators, by this meaning that there exists a polynomial matrix operator $R \in \mathbb{R}^{k \times q}$ such that

$$\mathcal{B} = \{ w \in \mathcal{H}^q : R \circ w = 0 \}. \quad (1)$$

It is always possible to find a matrix $\mathbf{R} \in \mathbb{R}^{r \times q}$ of full row rank $r$ such that $\mathcal{B} = \{ w \in \mathcal{H}^q : \mathbf{R} \circ w = 0 \}$.

A behavior $\mathcal{B} \subseteq \mathcal{H}^q$ is autonomous if it is a finite dimensional vector subspace of $\mathcal{H}^q$ as a vector space over $\mathbb{R}$ or $\mathbb{C}$. $\mathcal{B}$, described as in (1), is autonomous if and only if $R \in \mathbb{R}^{k \times q}$ is of full column rank $q$.

An autonomous behavior (1) is $I$-stable if the greatest common divisor of the maximal ($\geq qth$) order minors of $R$ is an $I$-stable polynomial. If $R$ is of full row rank and hence, under the autonomy assumption, nonsingular square, this amounts to requiring that $\text{det}(R)$ is $I$-stable. A trajectory $w \in \mathcal{H}^q$ is called $I$-small if it belongs to some autonomous $I$-stable behavior or, equivalently, if it satisfies the equation $p \circ w = 0$ for some $I$-stable polynomial $p$.

Clearly, for $a = 0$, $I = I_{c.a} = I_{s.a}$ corresponds to standard exponential stability of continuous-time linear differential systems, and the $I$-small signals in this case are the polynomial-exponential functions that converge to zero as the time tends to infinity. When $a > 0$, the choice $I = I_{c.a}$ imposes more stringent conditions on the exponential convergence to zero of the $I$-small trajectories, while $I = I_{s.a}$ imposes a bound on the oscillations of the complex modes exponentially converging to zero.

As different components of the behavior variable $w$ may have different meanings, it is often convenient to partition $w$ into blocks. This is the case, for instance, when we consider the interconnection of two different behaviors through a subset of their respective variables.

If $\mathcal{B}_1$ is a behavior with variable $\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \in \mathcal{F}^{q_1 + q_2}$ and $\mathcal{B}_2$ is a behavior with variable $\begin{pmatrix} w_1' \\ w_2' \end{pmatrix} \in \mathcal{F}^{q_1 + q_2}$, then we denote the partial interconnection of $\mathcal{B}_1$ and $\mathcal{B}_2$ via the components $w_2$ as follows:

$$\mathcal{B}_1 \otimes \mathcal{B}_2 := \left\{ \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \in \mathcal{F}^{q_1 + q_2 + q_3} : \begin{pmatrix} w_1' \\ w_2' \end{pmatrix} \in \mathcal{B}_1, \begin{pmatrix} w_2 \\ w_2' \end{pmatrix} \in \mathcal{B}_2 \right\}.$$

Given a behavior $\mathcal{B}$ with signals $\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \in \mathcal{F}^{q_1 + q_2 + q_3}$, we say that $w_1$ is $I$-observable from $w_2$ in $\mathcal{B}$ if the components $w_1$ are determined by $w_2$ up to an $I$-small signal, or, more precisely, if $\begin{pmatrix} w_1' \\ w_2' \end{pmatrix} \in \mathcal{B}$ and $w_2 = 0$ imply that $w_1$ is $I$-small.

If $\mathcal{B}$ is a behavior with variable $w = \begin{pmatrix} w_1' \\ w_2' \end{pmatrix} \in \mathcal{F}^{q_1 + q_2}$:

$$\mathcal{B} = \left\{ \begin{pmatrix} w_1' \\ w_2' \end{pmatrix} \in \mathcal{F}^{q_1 + q_2} : \begin{pmatrix} R_1, R_2 \end{pmatrix} \circ \begin{pmatrix} w_1' \\ w_2' \end{pmatrix} = 0 \right\},$$

where $(R_1, R_2) \in D^{k \times (q_1 + q_2)}$, we say that $w_2$ is free in $\mathcal{B}$ if for any $w_2 \in \mathcal{F}^{q_2}$ there exists $w_1 \in \mathcal{F}^{q_1}$ such that $\begin{pmatrix} w_1' \\ w_2' \end{pmatrix} \in \mathcal{B}$. This is the case iff \text{rank}(R_1, R_2) = \text{rank}(R_1)$. If additionally the behavior $\mathcal{B}^0 = \{ w_1 \in \mathcal{F}^{q_1} : R_1 \circ w_1 = 0 \}$ is autonomous, we say that $\mathcal{B}$ is an input/output behavior with input $w_2$ and output $w_1$. Clearly, this is the case iff \text{rank}(R_1, R_2) = \text{rank}(R_1) = q_1$. If the matrix $(R_1, R_2)$ is of full row rank $k$, it follows that $\mathcal{B}$ is an input/output behavior with input $w_2$ and output $w_1$ iff $k = q_1$ and $R_1$ is nonsingular square.

### III. The Consensus Problem

We assume that there are $N \geq 2$ homogeneous agents whose dynamics is described by the same behavior

$$\mathcal{P}_i := \mathcal{P} := \left\{ \begin{pmatrix} x_i \\ y_i \\ u_i \end{pmatrix} \in \mathcal{F}^{n+p+m} : (P_x, P_y, P_u) \circ \begin{pmatrix} x_i \\ y_i \\ u_i \end{pmatrix} = 0 \right\}$$

with $(P_x, P_y, P_u) \in D^{(n+p) \times (n+p+m)}$, $\det(P_x, P_u) \neq 0$, for $i = 1,\ldots,N$. The variables $x_i$ represent the target variables, i.e., those on which the agents should reach consensus. They are not necessarily directly measurable, and they do not necessarily represent state-variables, even if this is a possible case. The signals that can be measured (and are hence available for feedback interconnection) are the $y_i$'s. Note that the $\mathcal{P}_i$'s can be regarded as input/output behaviors with input $u_i$ and output $x_i$ and $y_i$. The assumption $\det(P_x, P_u) \neq 0$ ensures that the matrix $(P_x, P_y, P_u)$ is of full row rank $(n+p)$.

Upon setting $x := \begin{pmatrix} x_1' \\ \vdots \\ x_N' \end{pmatrix} \in \mathcal{F}^{Nn}$, $y := \begin{pmatrix} y_1' \\ \vdots \\ y_N' \end{pmatrix} \in \mathcal{F}^{Np}$, and $u := \begin{pmatrix} u_1' \\ \vdots \\ u_N' \end{pmatrix} \in \mathcal{F}^{Nm}$, the overall behavior of the $N$ agents can be described as follows:

$$\mathcal{P} := \{ \begin{pmatrix} x \\ y \\ u \end{pmatrix} : \begin{pmatrix} x \\ y \\ u \end{pmatrix} \in \mathcal{F}^{Nn+Np+Nm} : \forall i \in \{1,\ldots,N\}, \begin{pmatrix} x_i' \\ y_i' \\ u_i' \end{pmatrix} \in \mathcal{P}_i \}$$

$$= \{ \begin{pmatrix} x \\ y \\ u \end{pmatrix} : \begin{pmatrix} x \\ y \\ u \end{pmatrix} \in \mathcal{F}^{Nn+Np+Nm} :$$

$$\begin{pmatrix} I_N \otimes P_x, I_N \otimes P_y, I_N \otimes P_u \end{pmatrix} \circ \begin{pmatrix} x \\ y \\ u \end{pmatrix} = 0 \} \cong \mathcal{P}_1 \times \ldots \times \mathcal{P}_N,$$

where $\otimes$ denotes the Kronecker product of matrices, and $\cong$
means that \( \mathcal{P} \) and \( \mathcal{P}_1 \times \cdots \times \mathcal{P}_N \) are isomorphic, since their trajectories are related by a simple entry permutation.

The information flow between the agents is modeled [4], [12], [16] by a (directed or undirected) weighted graph \( \mathcal{G} \), with vertex set \( \mathcal{V} = \{1, \ldots, N\} \) (the \( i \)th vertex representing the \( i \)th agent) and edge set \( \mathcal{E} \subseteq \mathcal{V} \times \mathcal{V} \). The adjacency matrix of \( \mathcal{G} \) is denoted by \( A = (a_{ij})_{i,j \in \mathcal{V} \cap \mathcal{N} \subseteq \mathbb{R}^{N \times N} \) : \( a_{ij} \) is the weight of the edge from \( j \) to \( i \) if such an edge exists, and zero otherwise. The adjacency matrix gives rise to the Laplacian matrix \( L := \Delta \mathcal{G} \) with \( D := \text{diag}(A1_N) \).

Obviously, \( 1_N \) is an eigenvector of \( L \) corresponding to the zero eigenvalue. Let \( \text{spec}(L) = \{ \lambda_1, \ldots, \lambda_N \} \) denote the set of (not necessarily distinct and possibly complex) eigenvalues of \( L \). We assume that \( J \in \mathbb{C}^{N \times N} \) represents the Jordan normal form of the Laplacian matrix \( L \),

\[
J = \begin{pmatrix}
\lambda_1 & J_{1,2} & 0 & \cdots & 0 \\
0 & \lambda_2 & J_{2,3} & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & \lambda_{N-1} & J_{N-1,N} \\
0 & 0 & \cdots & 0 & \lambda_N
\end{pmatrix}.
\]

This implies that \( J_{i,i+1} \in \{0,1\} \), and \( \lambda_i = \lambda_{i+1} \) whenever \( J_{i,i+1} = 1 \). Moreover, let \( V \in \mathbb{C}^{N \times N} \) be a nonsingular transformation matrix, such that \( LV = VJ \). We assume that \( \lambda_1 = 0 \) and therefore the first column \( V_1 \) of \( V \) is an eigenvector of \( L \) corresponding to the zero eigenvalue. It entails no loss of generality assuming that \( V_1 = 1_N \).

Each \( i \)-th agent receives the weighted information

\[
\tilde{y}_i := \sum_{j=1}^{N} a_{ij}(y_i - y_j)
\]

from the other agents of the network, and designs a control strategy based on the signal \( \tilde{y}_i \). Specifically, we assume that the agents adopt (identical) controllers described by

\[
\mathcal{C}_i := \mathcal{C} := \left\{ \begin{pmatrix} \tilde{y}_i \\ u_i \end{pmatrix} \in \mathcal{F}^{n+m} : (C_y, C_u) \circ \begin{pmatrix} \tilde{y}_i \\ u_i \end{pmatrix} = 0 \right\}
\]

with \((C_y, C_u) \in \mathcal{G}^{k \times (p+m)} \), \( \text{rank}(C_u) = k \),

for \( i = 1, \ldots, N \). Hence the components \( \tilde{y}_i \) are free in the behaviors \( \mathcal{C}_i \). The controllers \( \mathcal{C}_i \) give rise to the overall controller

\[
\mathcal{K} := \left\{ \begin{pmatrix} \tilde{x} \\ \tilde{y} \\ u \end{pmatrix} \in \mathcal{F}^{n+m+n} : \forall i \in \{1, \ldots, N\}, \begin{pmatrix} \tilde{y}_i \\ u_i \end{pmatrix} \in \mathcal{C}_i \right\}
\]

\[
= \left\{ \begin{pmatrix} \tilde{x} \\ \tilde{y} \\ u \end{pmatrix} \in \mathcal{F}^{n+m+n} : \left( \begin{array}{c} I_N \otimes C_y \\ I_N \otimes C_u \end{array} \right) \ast \begin{pmatrix} \tilde{y} \\ u \end{pmatrix} = 0 \right\}
\]

\[
\cong \mathcal{C}_1 \times \cdots \times \mathcal{C}_N
\]

where, as above, \( \tilde{y} := \begin{pmatrix} \tilde{y}_1 \\ \vdots \\ \tilde{y}_N \end{pmatrix} \in \mathcal{F}^N \), \( u = \begin{pmatrix} u_1 \\ \vdots \\ u_N \end{pmatrix} \in \mathcal{F}^N \). The agents behaviors \( \mathcal{P}_1, \ldots, \mathcal{P}_N \), the graph \( \mathcal{G} \), and the controllers \( \mathcal{C}_1, \ldots, \mathcal{C}_N \) define the overall interconnected behavior

\[
\mathcal{K} := \left\{ \begin{pmatrix} x \\ y \\ u \end{pmatrix} \in \mathcal{F}^{n+m+n} : \forall i \in \{1, \ldots, N\}, \begin{pmatrix} x_i \\ y_i \\ u_i \end{pmatrix} \in \mathcal{P}_i, \begin{pmatrix} \tilde{y}_i \\ u_i \end{pmatrix} \in \mathcal{C}_i, \ \tilde{y}_i = \sum_{j=1}^{N} a_{ij}(y_i - y_j) \right\}
\]

\[
= \left\{ \begin{pmatrix} x \\ y \\ u \end{pmatrix} \in \mathcal{F}^{n+m+n} : \begin{pmatrix} I_N \otimes P_x \\ I_N \otimes P_y \end{array} \right) \circ \begin{pmatrix} x \\ y \\ u \end{pmatrix} = 0 \}
\]

This overall interconnection can also be interpreted as the partial interconnection \( \mathcal{K} = \mathcal{P} \wedge \mathcal{K}_L \) of the overall plant \( \mathcal{P} \) with the compensator

\[
\mathcal{K}_L := \left\{ \begin{pmatrix} y \\ u \end{pmatrix} \in \mathcal{F}^{n+m+n} : \left( \begin{array}{c} I_N \otimes C_y \\ I_N \otimes C_u \end{array} \right) \ast \begin{pmatrix} y \\ u \end{pmatrix} = 0 \right\}
\]

Definition 1. We say that the controllers \( \mathcal{C}_i = \mathcal{C}, \ i = 1, \ldots, N \), lead to consensus on \( x \) [or \( x \) and \( y \), resp.] among the \( N \) agents (described by \( \mathcal{P}_1, \ldots, \mathcal{P}_N \) and the graph \( \mathcal{G} \)) if, for every \( \begin{pmatrix} y \\ u \end{pmatrix} \in \mathcal{K} \), the deviations \( x_i - x \) are \( \mathcal{F} \)-small for all \( j \in \{2, \ldots, N\} \) [if the deviations \( x_i - x \) and \( y_j - y \) are \( \mathcal{F} \)-small for all \( j \in \{2, \ldots, N\} \), resp.].

Proposition 2. If there exist identical controllers \( \mathcal{C}_i = \mathcal{C}, \ i = 1, \ldots, N \), that lead to consensus on \( x \), then every variable \( x_i \) is \( \mathcal{F} \)-observable in \( \mathcal{P}_i = \mathcal{P} \) from the corresponding pair of variables \( \begin{pmatrix} y \\ u \end{pmatrix} \), \( i = 1, \ldots, N \).

Proof. For any \( i \neq 1 \), let \( \begin{pmatrix} 0 \\ 0 \end{pmatrix} \) be any trajectory in \( \mathcal{P}_i = \mathcal{P} \), and set \( x := (e_i \otimes I_{N}) x_i \), \( y := 0 \), and \( u := 0 \). Then \( \begin{pmatrix} y \\ u \end{pmatrix} \in \mathcal{K} \), and therefore, by the consensus property, \( x_i - x \) is \( \mathcal{F} \)-small. Being \( x_i = 0 \), this implies that \( x_i \) is \( \mathcal{F} \)-small. Clearly, as \( \mathcal{P}_i = \mathcal{P} \) for every \( i \neq 1 \), the same result applies to \( x_1 \).

In the following we will introduce the additional requirement that the feedback of \( \mathcal{P} \) and \( \mathcal{K}_L \) should be well-posed, i.e., the interconnection regular, according to Willems [20]. This means that the input/output structure of \( \mathcal{P} \) is preserved even after interconnection with \( \mathcal{K}_L \), in the sense that it is still possible to add (free) signals to the components of \( u \) after interconnection. More precisely, the components of \( u' \) have to be free in

\[
\left\{ \begin{pmatrix} x \\ y \\ u' \end{pmatrix} \in \mathcal{F}^{n+m+n+m} : \begin{pmatrix} x \\ y \\ u + u' \end{pmatrix} \in \mathcal{P}, \begin{pmatrix} y \\ u \end{pmatrix} \in \mathcal{K}_L \right\}
\]

as illustrated in Figure 1. For an interpretation of regularity for more general interconnections of behaviors (without pre-assigned input/output structure) see e.g. [2, Sec.VII].

The interconnection of two behaviors is regular if and only if the sum of the ranks of the two matrices appearing in
their kernel descriptions is equal to the rank of the matrix describing the interconnected behavior.

IV. Characterization of Controllers Leading to Consensus on $u$

In order to characterize the controllers that lead to consensus, it is convenient to introduce a variable transformation that transforms the overall connected system $\mathcal{K}$ into the following isomorphic (but in general complex) behavior:

$$\mathcal{K}_f := \left\{ \begin{pmatrix} x \\ y \\ \pi \end{pmatrix} \in \mathbb{F}^{Nn+Np+Nm} : \begin{pmatrix} I_N \otimes P_x & I_N \otimes P_y & I_N \otimes P_u \\ 0 & J \otimes C_y & I_N \otimes C_u \end{pmatrix} \circ \begin{pmatrix} x \\ y \\ \pi \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}. $$

As before, we denote $x = : (\pi_y \pi_y)$ and $y = : (\pi_y \pi_y) \in \mathbb{F}^{Np}$.

The behavior $\mathcal{K}_f$ can be interpreted as the (partial) interconnection $\mathcal{K}_f = \mathcal{K}_N \wedge \mathcal{K}_L$ of the overall plant $\mathcal{K}$ with the (complex) compensator $\mathcal{L}_L$.

See Figure 2 for an illustration of the structure of $\mathcal{K}_f$. For the sake of brevity, we introduce the notation

$$K := \begin{pmatrix} I_N \otimes P_x & I_N \otimes P_y & I_N \otimes P_u \\ 0 & L \otimes C_y & I_N \otimes C_u \end{pmatrix},$$

$$K_f := \begin{pmatrix} I_N \otimes P_x & I_N \otimes P_y & I_N \otimes P_u \\ 0 & J \otimes C_y & I_N \otimes C_u \end{pmatrix}.$$

Clearly, $\mathcal{K} = \mathcal{K}_N \wedge \mathcal{K}_L$ is the kernel of $K$, while $\mathcal{K}_f = \mathcal{K}_N \wedge \mathcal{L}_L$ is the kernel of $K_f$. By resorting to suitable row and column permutations, the matrix $K_f$ can be rewritten in the block triangular form

$$K_f^w := \begin{pmatrix} K_1 & K_{1,2} & 0 & \cdots & 0 \\ 0 & K_2 & K_{2,3} & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & K_{N-1} & K_{N-1,N} \\ 0 & 0 & \cdots & 0 & K_N \end{pmatrix},$$

where

$$K_1 := \begin{pmatrix} P_x & P_y & P_u \\ 0 & \lambda C_y & C_u \end{pmatrix} \in \mathbb{F}^{(n+p)\times(n+p+m)},$$

$$K_{i,i+1} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & J_{i,i+1} C_y & 0 \end{pmatrix} \in \mathbb{F}^{(n+p)\times(n+p+m)}. $$

Obviously, the variable transformation (or reordering of the components)

$$w_i := \begin{pmatrix} x_i \\ y_i \\ \pi_i \end{pmatrix}, \quad i = 1, \ldots, N, \quad w := \begin{pmatrix} w_1 \\ \vdots \\ w_N \end{pmatrix}$$

leads from $\mathcal{K}_f$ to the isomorphic behavior

$$\mathcal{K}_f^w := \left\{ w \in \mathbb{F}^{N(n+p)+m} : K_f^w \circ w = 0 \right\}.$$

**Lemma 3.** The behaviors $\mathcal{K}$ and $\mathcal{K}_f$ are related through the following isomorphism (variable transformation):

$$\begin{pmatrix} x \\ y \\ \pi \end{pmatrix} = \begin{pmatrix} V^{-1} \otimes I_n & 0 & 0 \\ 0 & V^{-1} \otimes I_p & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ \pi \end{pmatrix},$$

where $V$ has been defined in the previous section.

**Proof.** As $V \in \mathbb{C}^{N \times N}$ is nonsingular, it follows that also $V^{-1}$ and $V^{-1} \otimes I_p$ are invertible over $\mathbb{C}$ and hence in particular over $\mathcal{D} = \mathbb{C}[s]$. $V^{-1}$ obviously defines a $\mathcal{D}$-isomorphism from $\mathbb{F}^{N(n+p)+m}$ into $\mathbb{F}^{N(n+p)+m}$. We show that $\begin{pmatrix} \pi \\ \pi \end{pmatrix}$ is in $\mathcal{K}$ iff $\begin{pmatrix} \pi \\ \pi \end{pmatrix}$ is in $\mathcal{K}_f$. By definition, $\begin{pmatrix} \pi \\ \pi \end{pmatrix}$ is in $\mathcal{K}$ iff $K \circ \begin{pmatrix} \pi \\ \pi \end{pmatrix} = 0$, or equivalently iff

$$((V^{-1})^{-1} K V) \circ \begin{pmatrix} \pi \\ \pi \end{pmatrix} = 0,$$

namely $K_f \circ \begin{pmatrix} \pi \\ \pi \end{pmatrix} = 0$, i.e., iff

$$\begin{pmatrix} \pi \\ \pi \end{pmatrix} \in \mathcal{K}_f. \quad \Box$$

**Theorem 4.** The following statements are equivalent:

1) the interconnection $\mathcal{K} = \mathcal{K}_N \wedge \mathcal{L}_L$ is regular, and

2) the controllers $\mathcal{C}_i$, $i = 1, \ldots, N$, lead to consensus on $u$, i.e.,

$$\forall i \in \{2, \ldots, N\}, \quad x_i - x_1$$

are $\mathcal{I}$-small.

2) the interconnection $\mathcal{K}_f = \mathcal{K}_N \wedge \mathcal{L}_f$ is regular, and

3) the behavior $\mathcal{K}_f$ satisfies

$$\forall i \in \{2, \ldots, N\}, \quad \bar{x}_i$$

are $\mathcal{I}$-small.

**Proof.** 1a $\Leftrightarrow$ 2a: Condition 1a, i.e., regularity of the interconnection $\mathcal{K} = \mathcal{K}_N \wedge \mathcal{L}_L$, is equivalent to the matrix $K$ having full row rank, whereas condition 2a, i.e., regularity of $\mathcal{K}_f = \mathcal{K}_N \wedge \mathcal{L}_f$, is equivalent to $K_f$ having full row rank. The relationship $K_f = (V')^{-1} K V$ (with the invertible matrices $V$ and $V'$ from the proof of Lemma 3) implies that $K$ has full row rank if and only if $K_f$ does.
1b ⇔ 2b: Upon introducing the \((N-1) \times N\) matrix \(S := (-I_{N-1} \ I_{N-1})\), statement 1b can be rewritten as
\[
\forall \left( \frac{x}{y} \right) \in \mathcal{K}, \quad (S \otimes I_n)x \text{ is } \mathcal{J}\text{-small.}
\]

On the other hand, by Lemma 3, \(\left( \frac{x}{y} \right) \in \mathcal{K}\) if and only if \(\left( \frac{\tilde{x}}{\tilde{y}} \right) \in \mathcal{K}_j\), where \(x, \tilde{x}\) and \(y, \tilde{y}\) are related to \(x, u\) and \(y\) via equation (2). In particular, \(x = (V \otimes I_n)\tilde{x}\). So, condition 1b can be equivalently rewritten as
\[
\forall \left( \frac{\tilde{x}}{\tilde{y}} \right) \in \mathcal{K}_j, \quad (S \otimes I_n)(V \otimes I_n)\tilde{x} = (SV \otimes I_n)\tilde{x} \text{ is } \mathcal{J}\text{-small.}
\]

The fact that \(V\) is invertible over \(\mathbb{C}\) and the identity
\[
V = \left( \begin{array}{c|c|c|c}
1 & V_{12} & \cdots & V_{1N} \\
1 & V_{22} & \cdots & V_{2N} \\
\vdots & \vdots & \ddots & \vdots \\
1 & V_{N2} & \cdots & V_{NN} \\
\end{array} \right) = \left( \begin{array}{c|c|c|c}
1 & V_{12} & \cdots & V_{1N} \\
0 & V_{22} - V_{12} & \cdots & V_{2N} - V_{1N} \\
\vdots & \vdots & \ddots & \vdots \\
0 & V_{N2} - V_{12} & \cdots & V_{NN} - V_{1N} \\
\end{array} \right)
\]

imply that also
\[
\tilde{V} := \left( \begin{array}{c|c|c|c}
V_{12} - V_{22} & \cdots & V_{1N} - V_{2N} \\
\vdots & \ddots & \vdots \\
V_{N2} - V_{12} & \cdots & V_{NN} - V_{1N} \\
\end{array} \right) \in \mathbb{C}^{(N-1) \times (N-1)}
\]
is invertible over \(\mathbb{C}\). Since \(SV = (0, \tilde{V}) \in \mathbb{C}^{(N-1) \times (1+(N-1))}\), condition (3) is equivalent to
\[
\forall \left( \frac{\tilde{x}}{\tilde{y}} \right) \in \mathcal{K}_j, \quad ((0, \tilde{V}) \otimes I_n)\tilde{x} \text{ is } \mathcal{J}\text{-small,}
\]
and by the invertibility of \(\tilde{V}\) over \(\mathbb{C}\), this is equivalent to
\[
\forall \left( \frac{\tilde{x}}{\tilde{y}} \right) \in \mathcal{K}_j, \quad ((0, I_{N-1} \ - I_n)\tilde{x} \text{ is } \mathcal{J}\text{-small,}
\]
i.e., statement 2b is satisfied. \(\Box\)

Remark 5. A controller \(\mathcal{C}_j\) is called an \(\mathcal{J}\)-stabilizing controller for \(\mathcal{P}\) if the interconnection \(\mathcal{K} = \mathcal{P} \cap \mathcal{C}_j\) is regular and the interconnected system \(\mathcal{K} = \mathcal{P} \cap \mathcal{C}_j\) is an autonomous \(\mathcal{J}\)-stable behavior. Condition 2 in Theorem 4 means something weaker, namely that the controller \(\mathcal{C}_j\) is \(\mathcal{J}\)-stabilizing only for the variables \(x_i, i = 2, 3, \ldots, N\), of the complex plant \(\mathcal{P}\).

If we assume that the underlying graph structure is not arbitrary, the result provided by Theorem 4 can be strengthened. We address, specifically, the case when the Laplacian matrix \(L\) is diagonalizable, a situation that is always encountered when the graph \(\mathcal{G}\) is undirected, and hence the adjacency matrix is symmetric. This corresponds to the rather common case when the \(i\)th agent gives to the information received from the \(j\)th agent the same weight that the \(j\)th agent gives to the information received from the \(i\)th agent. In this situation, the behavior
\[
\mathcal{K}_j = \left\{ \left( \begin{array}{c} \tilde{x} \\ \tilde{y} \\ \tilde{u} \end{array} \right) \in \mathbb{R}^{3} : \forall i \in \{1, \ldots, N\}, \left( \begin{array}{c} \tilde{x}_i \\ \tilde{y}_i \\ \tilde{u}_i \end{array} \right) \in \mathcal{C}_i \right\}
\]

is the direct product of \(N\) independent behaviors, as illustrated in Figure 2, where the interconnections plotted in grey are now non-existent. Note that the behaviors \(\mathcal{P} \cap \mathcal{C}_i\) can be described as the kernels of \(K_i\). In analogy to \(\mathcal{C}_i\), we introduce the notation
\[
\mathcal{P}(\lambda_i) := \left\{ \left( \begin{array}{c} \tilde{x}_i \\ \tilde{y}_i \\ \tilde{u}_i \end{array} \right) \in \mathbb{R}^{3} : \left( \begin{array}{c} \tilde{x}_i \\ \tilde{y}_i \\ \lambda_i \tilde{u}_i \end{array} \right) \in \mathcal{C}_i \right\}.
\]

Corollary 6 (Diagonalizable Laplacian matrix). Assume that the Laplacian matrix \(L\) of the graph \(\mathcal{G}\) is diagonalizable, i.e.,
that $J = \text{diag}(\lambda_1, \ldots, \lambda_N)$ or that $J_{i,i+1} = 0$ for $i = 1, \ldots, N - 1$. Then the following statements are equivalent:

1) a) The interconnection $\mathcal{H}_i = \mathcal{P} \land \mathcal{C}_i$ is regular, and 
  b) the N controllers $\mathcal{G}_i = \mathcal{C}_i$, $i = 1, \ldots, N$, lead to consensus on $x$.

2) a) The interconnection $\mathcal{H}_i = \mathcal{P} \land \mathcal{C}_i$ is regular, and 
  b) $\forall \left( \begin{array}{c} x \\ y \\ \pi \end{array} \right) \in \mathcal{H}_i$, $\forall i \in \{2, \ldots, N\}$, $\pi_i$ is $\mathcal{I}$-small.

3) For every $i \in \{2, \ldots, N\}$
   a) the interconnection $\mathcal{P} \land \mathcal{C}(\lambda_i)$ is regular, and 
   b) $\forall \left( \begin{array}{c} x_i \\ y_i \\ \pi_i \end{array} \right) \in \mathcal{P} \land \mathcal{C}(\lambda_i)$, $\pi_i$ is $\mathcal{I}$-small.

4) For every $\lambda_i \in \text{spec}(L) \setminus \{0\}$
   a) the interconnection $\mathcal{P} \land \mathcal{C}(\lambda_i)$ is regular, and 
   b) $\forall \left( \begin{array}{c} \dot{x}_i \\ y_i \\ \hat{\pi}_i \end{array} \right) \in \mathcal{P} \land \mathcal{C}(\lambda_i)$, $\hat{x}_i$ is $\mathcal{I}$-small. Additionally, 
   c) if there exists $i \geq 2$ such that $\lambda_i = 0$, then also 
      $\forall \left( \begin{array}{c} x_i \\ y_i \\ \pi_i \end{array} \right) \in \mathcal{P} \land \mathcal{C}(0)$, $\pi_i$ is $\mathcal{I}$-small.

Proof. 1a $\Leftrightarrow$ 2a: It was already shown in Theorem 4.

2a $\Leftrightarrow$ 3a: For later application, we show this equivalence without using the assumption that $J$ is diagonal. The interconnection $\mathcal{H}_i = \mathcal{P} \land \mathcal{C}_i$ is regular iff $K_i$ has full row rank. Since $K_i^\pi$ is obtained from $K_i$ by simple row and column permutations, $K_i$ has full row rank iff $K_i^\pi$ does. It is clear that this is always the case if, for all $i \in \{1, \ldots, N\}$, the blocks $K_i$ have full row rank (i.e., the interconnections $\mathcal{P} \land \mathcal{C}(\lambda_i)$ are regular). Conversely, assume that $K_i^\pi$ has full row rank and choose $i \in \{1, \ldots, N\}$. Then there exists $j \in \{1, \ldots, N\}$ such that $\lambda_j = \lambda_i$ and either $j = N$ or $J_{j,i+1} = 0$ (and hence $K_{j,i+1} = 0$). In both cases, the block $K_j$ is the only non-zero block in the corresponding rows of the full row rank matrix $K_i^\pi$, which implies that $K_j = K_i$ has full row rank. The asserted equivalence follows since the interconnection $\mathcal{P} \land \mathcal{C}(0)$ is by construction always regular.

3a $\Leftrightarrow$ 4a: As just remarked, the interconnection $\mathcal{P} \land \mathcal{C}(\lambda_i)$ is automatically regular for $\lambda_i = 0$. Now assume $i \in \{2, \ldots, N\}$ with $\lambda_i \neq 0$. Then the regularity of $\mathcal{P} \land \mathcal{C}(\lambda_i)$ amounts to the matrix $K_i = \begin{pmatrix} P_x & P_y & P_u \\ 0 & \lambda_i C_y & C_u \end{pmatrix}$

having full row rank. On the other hand, the regularity of $\mathcal{P}(\lambda_i) \land \mathcal{C}$ means that the matrix 

$\tilde{K}_i := \begin{pmatrix} P_x & P_y & \lambda_i P_u \\ 0 & C_y & C_u \end{pmatrix}$

has full row rank. Since $\lambda_i \neq 0$, one easily sees that 

$\tilde{K}_i = \begin{pmatrix} \lambda_i & 0 \\ 0 & 1 \end{pmatrix} K_i \begin{pmatrix} 1/\lambda_i & 0 & 0 \\ 0 & 1/\lambda_i & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

Therefore the former matrix has full row rank if and only if the latter does.

1b $\Leftrightarrow$ 2b: It was already shown in Theorem 4.

2b $\Leftrightarrow$ 3b: It follows from the special form of $\mathcal{H}_j$ when $J$ is diagonal.

3b $\Leftrightarrow$ 4b+4c: For every $\lambda_i \in \text{spec}(L) \setminus \{0\}$, it can easily be checked that $\left( \begin{array}{c} \dot{y}_i \\ \dot{\hat{\pi}}_i \\ \dot{\pi}_i \end{array} \right) \in \mathcal{P} \land \mathcal{C}(\lambda_i)$ if and only if $\left( \begin{array}{c} \dot{y}_i \\ \dot{\hat{\pi}}_i \\ \dot{\pi}_i \end{array} \right) \in \mathcal{P} \land \mathcal{C}(\lambda_i)$. Consequently, $\pi_i$ is an $\mathcal{I}$-small trajectory for all $\left( \begin{array}{c} \dot{y}_i \\ \dot{\hat{\pi}}_i \\ \dot{\pi}_i \end{array} \right) \in \mathcal{P} \land \mathcal{C}(\lambda_i)$ if and only if $\hat{x}_i$ is an $\mathcal{I}$-small trajectory for all $\left( \begin{array}{c} \dot{x}_i \\ \dot{y}_i \\ \dot{\pi}_i \end{array} \right) \in \mathcal{P}(\lambda_i) \land \mathcal{C}$. This proves that the conditions given in 3b for $\lambda_i \neq 0$ are equivalent to 4b. On the other hand, condition 4c is just condition 3b for $\lambda_i = 0$.

Rem 7. Condition 4 in the previous result states that any controller $\mathcal{C}$ that leads to consensus must simultaneously $\mathcal{I}$-stabilize the variable $\hat{x}_i$ in the plants $\mathcal{P}(\lambda_i)$ for every $\lambda_i \in \text{spec}(L) \setminus \{0\}$.

If $\lambda_1 = 0$ is not the only zero eigenvalue of the Laplacian matrix $L$, then condition 4c implies that a necessary condition for solvability of the consensus problem is that $\pi_i$ has to be $\mathcal{I}$-small whenever $\dot{\pi}_i \in \mathcal{P}$ and $\pi_i = 0$. This means that $\pi_i$ has to be $\mathcal{I}$-observable from $\pi_i$ alone in $\mathcal{P}$, and not only from $\left( \begin{array}{c} \gamma_i \\ \pi_i \end{array} \right)$ as guaranteed by Proposition 2.

Rem 8. In the general case, when $L$ is not diagonalizable, consensus on $x$ for the overall interconnected system is not guaranteed by condition 3 of the previous Proposition, namely by regularity of the interconnection and the property

$\forall i \in \{2, \ldots, N\}, \forall \left( \begin{array}{c} x_i \\ y_i \\ \pi_i \end{array} \right) \in \mathcal{P} \land \mathcal{C}(\lambda_i), \quad \pi_i$ is $\mathcal{I}$-small,

i.e., the results derived in Corollary 6 do not extend to generic matrices $L$. This fact is enlightened by the following example.

We choose the stability region $\mathcal{I}$ so that nonzero constant signals are not $\mathcal{I}$-small (e.g., we may choose $\mathcal{I} = \mathcal{I}_{c,0}$) and hence obtain that $\mathcal{I}$-small signals are the polynomial-exponential ones that converge to zero as $t$ goes to $+\infty$. Assume that there are $N = 3$ agents, whose dynamics is described by $\mathcal{P}_i := \mathcal{P}$ with

$\mathcal{P} := \left\{ \left( \begin{array}{c} x_i \\ y_i \\ u_i \end{array} \right) \in \mathcal{F}^{1+1+1} : \left( \begin{array}{ccc} 1 & 1 & -1 \\ 0 & s & 0 \end{array} \right) \circ \left( \begin{array}{c} x_i \\ y_i \\ u_i \end{array} \right) = 0 \right\}$,

$s$ representing the time derivative, and the interactions among the agents are described by the Laplacian matrix $L$ with Jordan form $J$ where

$L := \begin{pmatrix} 2 & -1 & -1 \\ -1 & 3 & -2 \\ -2 & -1 & 3 \end{pmatrix}$, \quad $J := \begin{pmatrix} 0 & 0 \\ 0 & 4 \\ 0 & 0 \end{pmatrix}$,

and hence $\lambda_1 = 0, \lambda_2 = \lambda_3 = 4$. Furthermore, consider the controllers

$\mathcal{G}_i := \mathcal{C} := \left\{ \left( \begin{array}{c} \hat{y}_i \\ u_i \end{array} \right) \in \mathcal{F}^{1+1} : \lambda_i \hat{y}_i = u_i \right\}, \quad i = 1, 2, 3.$
It is easy to verify that
\[ \forall \begin{pmatrix} x_i \\ y_i \\ u_i \end{pmatrix} \in \mathcal{P} \land \mathcal{C}(\lambda_2) = \mathcal{P} \land \mathcal{C}(\lambda_3), \quad x_i \text{ is } \mathcal{I} \text{-small.} \]

This is simply due to the fact that a trajectory \( \begin{pmatrix} x_i \\ y_i \end{pmatrix} \) belongs to \( \mathcal{P} \land \mathcal{C}(\lambda_2) \) if and only if \( x_i + y_i = \pi_i, \) \( s \circ y_i = 0, \) and \( y_i = \pi_i, \) but then \( x_i \) is necessarily zero. Also, the interconnection \( \mathcal{P} \land \mathcal{C}(\lambda_2) \) is regular. However, the controllers \( \mathcal{C}_i \) do not lead to consensus on \( x. \) To see that, by Theorem 4, it is sufficient to show that there exists \( \begin{pmatrix} x_i \\ y_i \end{pmatrix} \in \mathcal{X}_i \) such that either \( x_i \) or \( y_i \) is not \( \mathcal{I} \)-small. It is a matter of simple computations to verify that the signal \( \begin{pmatrix} x_i \\ y_i \end{pmatrix} \) obtained from the blocks
\[
\begin{pmatrix} \ddot{x}_1 \\ \ddot{y}_1 \\ \pi_1 \end{pmatrix} := \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} \ddot{x}_2 \\ \ddot{y}_2 \\ \pi_2 \end{pmatrix} := \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} \ddot{x}_3 \\ \ddot{y}_3 \\ \pi_3 \end{pmatrix} := \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}
\]

indeed belongs to \( \mathcal{X}_i, \) but obviously \( \ddot{x}_2 = 1 \) is not \( \mathcal{I} \)-small.

V. CONSENSUS ON BOTH \( x \) AND \( y \)

In the case where consensus is not only required on the components \( x_i, \) but also on the \( y_i \)’s, it is possible to refine the results of Theorem 4 even without the assumption that \( L \) is diagonalizable.

Proposition 9 (Consensus on \( x \) and \( y \)). The following statements are equivalent:

1) a) The interconnection \( \mathcal{H} = \mathcal{P} \land \mathcal{C}_L \) is regular, and

b) the controllers \( \mathcal{C}_i \) lead to consensus on \( x \) and \( y, \)

i.e., \( \forall \begin{pmatrix} x_i \\ y_i \end{pmatrix} \in \mathcal{H}, \forall i \in \{2, ..., N\}, \quad x_i - x_1 \text{ and } y_i - y_1 \) are \( \mathcal{I} \)-small.

2) a) The interconnection \( \mathcal{H}_i = \mathcal{P} \land \mathcal{C}_i \) is regular, and

b) \( \forall \begin{pmatrix} x_i \\ y_i \end{pmatrix} \in \mathcal{H}_i, \forall i \in \{2, ..., N\}, \quad x_i \text{ and } y_i \) are \( \mathcal{I} \)-small.

3) For every \( i \in \{2, ..., N\}, \)

a) the interconnection \( \mathcal{P} \land \mathcal{C}(\lambda_i) \) is regular, and

b) \( \forall \begin{pmatrix} x_i \\ y_i \end{pmatrix} \in \mathcal{P} \land \mathcal{C}(\lambda_i), \quad x_i \text{ and } y_i \) are \( \mathcal{I} \)-small.

Due to page constraints, the proof is omitted.

Remark 10. It is worthwhile to follow up on Remark 8, at the end of Section 3. In that remark we underlined that, in the general case, when \( L \) is not diagonalizable, it is not true that condition:

\[ \forall i \in \{2, ..., N\}, \forall \begin{pmatrix} x_i \\ y_i \\ u_i \end{pmatrix} \in \mathcal{P} \land \mathcal{C}(\lambda_i), \quad x_i \text{ is } \mathcal{I} \text{-small} \]

ensures consensus on \( x \) for the overall connected system. What clearly arises from Proposition 9, instead, is that if we search for consensus on both \( x \) and \( y \) and we look for a regular interconnection, then condition 3b becomes equivalent to consensus on \( x \) and \( y. \)

Note that the previous result remains valid also if consensus, via regular interconnection, is formally required only on \( x, \) but the structure of \( \mathcal{P} \) and \( \mathcal{C} \) is such that consensus on \( x \) necessarily ensures also consensus on \( y. \) This is the case in most of the situations investigated in the literature, where the structure of \( \mathcal{P} \) and \( \mathcal{C} \) is normally much more restrictive than in the present approach (see also Section VI).

VI. COMPARISON WITH SOME RESULTS FROM THE LITERATURE

In this section we aim at comparing the results derived in the previous sections with some results available in the literature. In the following examples, we choose \( \mathcal{I} = \mathcal{I}_{0,0}, \) and hence the \( \mathcal{I} \)-small signals are the polynomial-exponential functions that converge to zero, and the \( \mathcal{I} \)-stable behaviors are the autonomous behaviors that are asymptotically (Hurwitz) stable. We use the same notation as in the previous sections.

Example 11. In [16] each agent dynamics takes the form
\[
\mathcal{P}_i := \mathcal{P} := \left\{ \begin{pmatrix} x_i \\ u_i \end{pmatrix} \in \mathbb{R}^{n+m} : \dot{x}_i = Ax_i + Bu_i \right\}
\]

\[= \left\{ \begin{pmatrix} x_i \\ u_i \end{pmatrix} \in \mathbb{R}^{n+m} : (sI_n - A, -B) \circ \begin{pmatrix} x_i \\ u_i \end{pmatrix} = 0 \right\}
\]

for constant matrices \( A \in \mathbb{R}^{n \times n} \) and \( B \in \mathbb{R}^{n \times m}, \) and the network graph is a directed weighted graph. The information available for the \( i \)-th agent is

\[ \ddot{x}_i := \sum_{j=1}^{N} a_{ij}(x_j - x_i) \]

(which differs from the present set-up because it is a state feedback instead of an output feedback, and it also differs in a sign). The considered controllers are of the form

\[ \mathcal{C}_i := \mathcal{C} := \left\{ \begin{pmatrix} x_i \\ u_i \end{pmatrix} \in \mathbb{R}^{n+m} : u_i = Kx_i \right\} \]

for a constant matrix \( K \in \mathbb{R}^{m \times n}. \) Lemma 3.1 in [16], which is quoted from [4], [17], states that consensus on \( x \) is reached if and only if the matrices \( A - \lambda_i BK, i \in \{2, ..., N\}, \) are Hurwitz. This is equivalent to the condition

\[ \forall i \in \{2, ..., N\}, \quad \forall \begin{pmatrix} x_i \\ u_i \end{pmatrix} \in \mathcal{P} \land \mathcal{C}(\lambda_i), \quad x_i \text{ is } \mathcal{I} \text{-small} \]

since
\[
\mathcal{P} \land \mathcal{C}(\lambda_i) = \left\{ \begin{pmatrix} x_i \\ u_i \end{pmatrix} \in \mathbb{R}^{n+m} : (sI_n - A) \circ x_i = B \circ u_i, \right\}
\]

\[= \left\{ \begin{pmatrix} x_i \\ u_i \end{pmatrix} \in \mathbb{R}^{n+m} : (sI_n - (A - \lambda_i BK)) \circ x_i = 0, \right\}
\]

\[= \left\{ \begin{pmatrix} x_i \\ u_i \end{pmatrix} \in \mathbb{R}^{n+m} : (sI_n - (A - \lambda_i BK)) \circ x_i = 0, \right\}
\]

Taking into account the different definition of \( \ddot{x}_i \) leading to the minus sign in \( \mathcal{C}(\lambda_i), \) and the fact that in [16] the variables \( x_i \) and \( y_i \) coincide, this result is in agreement with Proposition 9.
Example 12. In [5] agent dynamics are of the form

\[ \mathcal{P}_i := \mathcal{P} := \left\{ \begin{pmatrix} x_i \\ y_i \\ u_i \end{pmatrix} \in \mathbb{R}^{n+p+m} : \left( \begin{array}{cc} sI_n - A & 0 \\ -C & I_p \end{array} \right) \circ \begin{pmatrix} x_i \\ y_i \end{pmatrix} = \begin{pmatrix} B \\ 0 \end{pmatrix} \circ u_i \right\} \]

for constant matrices \( A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times m} \) and \( C \in \mathbb{R}^{p \times n} \), and the network graph is undirected and unweighted. Moreover, it is assumed that zero is a simple eigenvalue of \( L \), i.e., that \( \lambda_i \neq 0 \) for \( i \in \{2, \ldots, N\} \). Each \( i \)th agent can apply a controller with the input

\[ \tilde{y}_i := \sum_{j=1}^{N} a_{ij}(y_i - y_j). \]

The studied controllers are of the form

\[ \mathcal{C}_i := \mathcal{C} := \left\{ \begin{pmatrix} \tilde{y}_i \\ u_i \end{pmatrix} \in \mathbb{R}^{p+m} : u_i = K \tilde{y}_i \right\} \]

for a constant matrix \( K \in \mathbb{R}^{m \times p} \). According to Proposition 1 in [5], consensus is reached if and only if, for all nonzero eigenvalues \( \lambda_i \) of the graph Laplacian matrix \( L \), the system

\[ \begin{cases} \dot{x}_i = Ax_i + Bu_i, \\ \dot{y}_i = Cy_i, \\ \dot{u}_i = \lambda_i K \tilde{y}_i \end{cases} \]

is asymptotically stable. This equivalence does also follow from Corollary 6 or Proposition 9 in the present paper.

VII. CONCLUDING REMARKS

In this paper we have presented some preliminary results about the consensus problem for \( N \) homogeneous agents whose dynamics are described by the same behavioral model. Necessary and sufficient conditions for a group of homogeneous dynamic controllers, making use of the weighted information exchanged by the agents, to allow consensus among the agents (either on the target variables or on both the target and the measurable variables) are presented. Future research should aim at finding necessary and sufficient conditions on the agents’ dynamics and on the adjacency matrix for the existence of such homogenous dynamic controllers. Also, a parametrization of such controllers should be derived, thus allowing to choose among them the ones that guarantee best performances. Finally, the previous problems should be addressed for the case when the Laplacian matrix is not diagonalizable.

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