Moving Horizon Estimation with Pre-Estimation (MHE-PE) for 3D Space Debris Tracking during Atmospheric Re-entry

Rata Suwantong¹, Sylvain Bertrand¹, Didier Dumur² and Dominique Beauvois²

Abstract—Space debris tracking during atmospheric re-entry is a very complex problem due to high variations with time of the ballistic coefficient. The nature of these variations is generally unknown and an assumption has to be made in the estimation model which can result in high model errors. An estimator which is robust against model errors is therefore required. In previous work done by the authors, Moving Horizon Estimation (MHE) has been shown to outperform other classical nonlinear estimators in terms of accuracy and robustness against poor initialization for a simplified 1D case of space debris tracking during the re-entry. However, the large computation time of the MHE prevents its implementation for the 3D cases. Recently, the Moving Horizon Estimation with Pre-Estimation (MHE-PE) which requires much less computation time than the classical MHE while keeping its accuracy and robustness has been proposed. This paper therefore implements the MHE-PE to solve the 3D space debris tracking problem during the re-entry. Its performances are compared to some classical nonlinear estimators in terms of non-divergence percentage, accuracy and computation time through Monte Carlo simulations.

I. INTRODUCTION

Existing works on ballistic object tracking during the atmospheric re-entry concern mainly military vehicles whose ballistic coefficient \( \beta \), representing the object’s ability to overcome the drag, is nearly constant [4][13]. In this case, an estimation model assuming that \( \beta \) is driven by a zero-mean Gaussian white noise is generally used [13]. Tracking space debris during atmospheric re-entry is a more complex problem because \( \beta \) can have high variations with time. Moreover, the evolution of \( \beta \) in case of space debris is generally unknown. An assumption on its variation has to be made in the estimation model which results in model errors. In the previous work by the authors [17], we showed that, for a simplified 1D space debris tracking during the re-entry, classical estimators such as the Extended Kalman estimator (EKF), the Unscented Kalman estimator (UKF) and the Regularized Particle estimator (RPF) are outperformed by the Moving Horizon Estimator (MHE) in terms of robustness against poor initialization and accuracy. However, the MHE requires very large computation time and its implementation in 3D cases is limited.

The estimates of the MHE are obtained by solving an optimization problem at each instant [2]. Using a finite number of latest measurements over a given horizon, the MHE computes the state at the beginning of the horizon and the sequence of the process noise over the horizon that minimize a cost function. This cost function takes into account norm of the difference between real and predicted measurements over the horizon, process noise over the horizon and norm of the difference between the initial state at the beginning of the horizon and an a priori initial state. In the MHE, the evolution of the state over the horizon is constrained by the state equation of the estimation model. The MHE allows to handle nonlinear models and incorporate constraints directly. It has been shown to be robust against poor initialization [6]. However, its computation time can be very large due to its high number of optimization parameters.

To reduce computation time, fast optimization techniques for the MHE have been proposed in [5][19]. An MHE strategy to reduce computation time without changing optimization method was proposed in [14] for discrete-time linear systems, based on an introduction of a pre-estimating observer. An extension to discrete-time nonlinear systems under bounded noise has been proposed in [16]. In this strategy, called a Moving Horizon Estimator with Pre-Estimation (MHE-PE), the state over the horizon is propagated using the equation of an auxiliary estimator, called pre-estimator, instead of the state equation as in the classical MHE. This pre-estimator helps to compensate for model errors without having to estimate the optimal process noise sequence over the horizon. The initial state of the horizon hence becomes the only optimization parameter. The MHE-PE has shown to require smaller computation time compared to the MHE, while keeping its robustness and accuracy advantages [16]. This method suits, therefore, for the estimation problems for which the computation time of the MHE would be prohibitive such as space debris tracking during the re-entry.

In this paper, the 3D space debris tracking problem will be addressed using the MHE-PE. The estimation errors, non-divergence percentage and computation time of the MHE-PE will be compared to those of the EKF, the UKF and the RPF through Monte Carlo simulations. Compared to previous work [17], a new estimation model using the derivative of the acceleration is proposed in this paper.

The paper is structured as follows: In section II, the real system, the measurement equation and the estimation model are presented. In section III, the formulation of the MHE-PE is introduced. In section IV, the performances of the estimators are studied, followed by concluding remarks.
II. REAL SYSTEM AND ESTIMATION MODEL

A. Dynamics of the real system

Consider the trajectory of a piece of space debris during atmospheric reentry. Assume that the only two major forces acting on the object are the gravitational force and the drag and that the Earth’s rotation is neglected due to the short period of the reentry. The dynamics of the object is therefore described in the Earth-centered, Earth fixed (ECEF) coordinates. Denote \( \mathbf{r}_{\text{EF}} = \hat{O}\mathbf{D} \in \mathbb{R}^3 \), \( \mathbf{v}_{\text{EF}} \in \mathbb{R}^3 \) and \( \mathbf{a}_{\text{EF}} \in \mathbb{R}^3 \) the position, the velocity and the acceleration of the object in the ECEF coordinates, see fig. 1, and \( H \) its altitude. The dynamics of the object is described by

\[
\mathbf{a}_{\text{EF}}(t) = -\frac{GM}{||\mathbf{r}_{\text{EF}}(t)||^3} \mathbf{r}_{\text{EF}}(t) - \frac{\rho(H(t))}{2B(t)} ||\mathbf{v}_{\text{EF}}(t)|| \mathbf{v}_{\text{EF}}(t) \tag{1}
\]

where \( G \) is the Earth’s gravitational constant and \( M \) is the mass of the Earth. The atmosphere density \( \rho \) depends on the altitude of the object \( H(t) = ||\mathbf{r}_{\text{EF}}(t)|| - R_E \) where \( R_E \) is the mean Earth radius. The atmosphere density \( \rho \) is modeled by \( \rho(H) = c_1 e^{-c_2 H} \) where \( c_1 = 1.227 \text{ kg/m}^3 \), \( c_2 = 1.093 \times 10^{-4} \text{ m}^{-1} \) for \( H < 9144 \text{ m} \) and \( c_1 = 1.754 \text{ kg/m}^3 \), \( c_2 = 1.490 \times 10^{-4} \text{ m}^{-1} \) for \( H \geq 9144 \text{ m} \). \( B \in \mathbb{R}^{+} \) is the object’s ballistic coefficient defined by \( \beta = A \cdot C_D \) where \( m, A \) and \( C_D \) are the mass, the cross section and the drag coefficient of the object respectively, \( \beta \) represents the body’s ability to overcome the drag. For space debris, \( C_D \) can depend on the altitude \( H(t) \) and the velocity \( \mathbf{v}(t) \). \( m \) and \( D \) also can vary with time if the object is burnt up and rotates. Therefore, \( \beta \) is a time varying function and its analytical expression is generally unknown.

B. Measurement Equations

The trajectory of the object will be estimated from noisy radar measurements which are the measurements of the distance \( d \), the elevation angle \( el \) and the azimuth angle \( az \) between the radar station and the object, supposed to be available every \( T_s \) second.

Denote \( \lambda \) and \( \phi \) the longitude and the latitude of the radar station respectively. Consider the local coordinates centered at the radar station with axes directed along the south, east and up (SEU) directions (fig. 1). Denote the position of the object in the SEU coordinates as \( \mathbf{r}(t) = \hat{S}\mathbf{D} \in \mathbb{R}^3 \) where \( \hat{S} \) is the radar station, \( \mathbf{D} \) the space debris, \( O \) the center of the Earth and \( \mathbf{E} \) the position of the object in the ECEF coordinates \( \mathbf{E} \). The position of the object in the SECEF coordinates \( \mathbf{r}_{\text{EF}} \) can be converted into the SEU coordinates \( \mathbf{r} \) using:

\[
\mathbf{r} = \begin{pmatrix}
\sin\phi\cos\lambda & \sin\phi\sin\lambda & -\cos\lambda \\
\cos\phi\cos\lambda & \cos\phi\sin\lambda & 0 \\
\cos\lambda & -\sin\lambda & 0
\end{pmatrix} \mathbf{r}_{\text{EF}} - \begin{pmatrix} 0 \\ 0 \\ R_E \end{pmatrix} \tag{2}
\]

Denote \( \mathbf{v} \) and \( \mathbf{a} \) the velocity and the acceleration of the object in the SEU coordinates respectively. They can be calculated from \( \mathbf{v}_{\text{EC}} \) and \( \mathbf{a}_{\text{EC}} \) using the transformation matrix in (2).

Define \( \mathbf{r}_k = \mathbf{r}(t = kT_s) \), \( k \in \mathbb{N} \), the discrete-time measurement vector at instant \( k \) as:

\[
y_k = \begin{pmatrix}
\mathbf{r}_k^m \\
\mathbf{e}_k^m \\
\mathbf{a}_k^m
\end{pmatrix} = \begin{pmatrix}
\mathbf{d}_k \\
\mathbf{e}_k \\
\mathbf{a}_k
\end{pmatrix} + \eta_k = \begin{pmatrix}
\sqrt{r_{kx}^2 + r_{ky}^2 + r_{kz}^2} \\
\arcsin\left(\frac{r_{kx}}{\sqrt{r_{kx}^2 + r_{ky}^2 + r_{kz}^2}}\right) \\
\arctan\left(\frac{r_{ky}}{r_{kx}}\right) + \eta_k
\end{pmatrix} \tag{3}
\]

\( \eta_k \) is a discrete-time measurement noise modelled by a zero-mean bounded white noise with covariance matrix \( \mathbf{R} = \text{diag}\left(\sigma^2_d, \sigma^2_{el}, \sigma^2_{az}\right) \). \( \sigma_d, \sigma_{el} \) and \( \sigma_{az} \) are the standard deviations (std) associated to the measurements of \( d, el \) and \( az \) respectively.

C. Estimation Model

The problem consists in estimating \( \mathbf{r} \) and \( \mathbf{v} \) of the object in the local SEU coordinates from the noisy radar measurements. Define \( \mathbf{SEU} \) coordinates as the local coordinates centered at the debris with axes directed along the south, east, up directions (fig. 1). It is assumed that debris is closed enough to the station that the Earth can be considered flat in the estimation model. Consequently, the unit vectors of the coordinates \( \mathbf{SEU} \) are parallel to those of the coordinates \( \mathbf{SEU} \). In this case, \( H \sim r_z \) and the gravity can be considered constant \( g = 9.81 \text{ m/s}^2 \) pointing downward.

In the estimation model, we choose to include the acceleration of the object, which is linked to \( \beta, \mathbf{r} \) and \( \mathbf{v} \). The acceleration in the estimation model is described as:

\[
\dot{\mathbf{r}}(t) = \mathbf{v}(t) = \mathbf{a}(t) = \begin{pmatrix}
-c_1 e^{-c_2 r_z} ||\mathbf{v}\|| \mathbf{v} \\
-c_1 e^{-c_2 r_z} ||\mathbf{v}\|| \mathbf{y} \\
-g - c_1 e^{-c_2 r_z} ||\mathbf{v}\|| \mathbf{v}
\end{pmatrix} \tag{4}
\]

To implement an estimator, a discrete-time version of the estimation model will be used. The sampling period of the estimator is chosen equal to \( T_s \) for the sake of simplicity. Define the state \( x(t) = (\mathbf{r}(t))^T \mathbf{v}(t)^T \mathbf{a}(t)^T \) and \( x_k = x(t = kT_s) \). It is assumed \( \forall t \in [kT_s, (k+1)T_s] \) that:

\[
\dot{\mathbf{a}}(t) = \mathbf{f}_k + \xi(t), \text{ where } \mathbf{f}_k = \mathbf{a}(t)_{t = kT_s} \tag{5}
\]

\( \xi(t) \) is a continuous zero-mean bounded white noise of spectral density \( \tilde{q} \) representing errors from discretization and the fact that \( \beta \neq \text{cst} \). Denote \( \mathbf{f}_k \simeq (f_{x_k}, f_{y_k}, f_{z_k})^T \) we have:

\[
f_{x_k} = -c_2 y_{z_k} + a_{y_k} \left( \frac{1}{||\mathbf{v}_k||} + v_{y_k} \right) + a_{y_k} v_{y_k} \left( \frac{1}{||\mathbf{v}_k||} + \frac{v_{y_k}}{||\mathbf{v}_k||} \right) \tag{6}
\]

\[
f_{y_k} = -c_2 y_{z_k} + a_{y_k} v_{y_k} \left( \frac{1}{||\mathbf{v}_k||} + v_{y_k} \right) + a_{y_k} v_{y_k} \left( \frac{1}{||\mathbf{v}_k||} + \frac{v_{y_k}}{||\mathbf{v}_k||} \right) \tag{7}
\]
$$ f_k = \left( -c_2 v_k + \left( a_{22} v_k + a_{23} v_k + a_{33} \left( \frac{1}{v_k^4} + \frac{v_0}{v_k^2} \right) \right) \right) (a_{31} \pm R) $$

Similarly to [3] the following state equation is derived for the estimation model:

$$ x_{k+1} = \left( \begin{array}{ccc} 1 & T_s & \frac{T_s^2}{2} \\ 0 & 1 & T_s \\ 0 & 0 & 1 \end{array} \right) x_k + f_k T_s + w_k $$

where $w_k$ is a discrete-time zero-mean bounded white noise whose covariance matrix is

$$ Q = \left( \begin{array}{ccc} \frac{1}{20} T_s^2 I_3 & \frac{1}{8} T_s^3 I_3 & \frac{1}{6} T_s^3 I_3 \\ \frac{1}{8} T_s^3 I_3 & \frac{1}{3} T_s^3 I_3 & \frac{1}{2} T_s^3 I_3 \\ \frac{1}{6} T_s^3 I_3 & \frac{2}{3} T_s^3 I_3 & T_s I_3 \end{array} \right) q $$

The measurement equation in (3) can also be written as:

$$ y_k = h(x_k) + \eta_k, y \in Y \subset \mathbb{R}^{n_y} $$

where the definition of $h$ is straightforward from (2) and (3).

In this study, it is supposed that 1. $X$ is a convex compact set, 2. $w_k \in \mathbb{W}$ and $\eta_k \in \mathbb{V}$ where $\mathbb{W}$ and $\mathbb{V}$ are compact sets containing 0 and 3. the initial state $x_0$ is such that, for any possible value of process noise $w_k$, $x_k \in X$. Based on this model, several estimators are implemented and compared to the proposed MHE-PE approach, presented below.

**III. MOVING HORIZON ESTIMATOR WITH PRE-ESTIMATION (MHE-PE)**

As mentioned in the introduction, the Moving Horizon Estimator with Pre-Estimation (MHE-PE) was proposed by the authors in [16] for discrete-time nonlinear systems under bounded noise to reduce computation time of the classical MHE. Consider first the formulation of the MHE-PE.

**A. Formulation of the MHE-PE**

Assume that the state vector $x_k$ has to be estimated at time step $k \geq N$ using the latest $N + 1$ measurements collected within a “sliding horizon” $[k-N,k]$. Let us denote $y_{k-N}^k = (y_{k-N}^T \ y_{k-N+1}^T \ \ldots \ y_k^T)^T$ the measurement collection over the horizon, $\hat{x}_{k-N|k}$ an estimate of $x_{k-N}$ computed at time step $k$ and $\hat{x}_{k-N|k}$ an a priori value of $\hat{x}_{k-N|k}$.

The estimate provided by the MHE-PE is obtained by solving the following optimization problem for $k \geq N$ [16]:

$$ \min_{\hat{x}_{k-N|k} \in \hat{X}} J_k(\hat{x}_{k-N|k}, y_{k-N|k}, y_{k-N}^k) $$

$$ J_k = \mu \|\hat{x}_{k-N|k} - \hat{x}_{k-N|k}^*\|^2 + \sum_{i=0}^N \|y_{k-N+i} - h(\hat{x}_{k-N+i|k})\|^2_R $$

s.t. $\hat{x}_{k-N+i+1|k} = g(\hat{x}_{k-N+i|k}, y_{k-N+i})$, $\forall i \in [0,N-1]$ (10c)

where $\mu$ is a positive scalar weight term representing the confidence in the a priori value $\hat{x}_{k-N|k}$ with respect to the observation model. $R$ is the measurement noise covariance matrix. $X$ is a set of authorized values of the estimate that can be related to physical constraints. Denote $\hat{x}_{k-N|k}^*$ the solution of the optimization problem (10). The estimate $\hat{x}_{k|k}$ of $x_k$ at instant $k$ provided by the MHE-PE is computed using (10c) from $\hat{x}_{k-N|k}^*$ and $y_{k-N+1}^k$. $\hat{x}_{k-N|k}$ is determined from $\hat{x}_{k-N|k}$.

We note that in the MHE-PE the evolution of the state at each instant over the horizon is subject to the equation of the estimator $g$ via (10c) instead of the state equation as in the classical MHE [2]. In fact, the MHE-PE implements the estimator $g$ locally over the horizon while $g$ is initialized at the beginning of the horizon by optimizing the cost function (10b) taking into account $N + 1$ latest measurements.

**B. Conditions on the pre-estimator $g$**

Let us define the observation maps of the estimator by

$$ G(\hat{x}_{k-N}, y_{k-N}^{k-1}) = \begin{pmatrix} h(\hat{x}_{k-N}) \\ \vdots \\ h(g^{N}(\hat{x}_{k-N}, y_{k-N}^{k-1})) \end{pmatrix} $$

where $g^{i}(\hat{x}_{k-N}, y_{k-N}^{k-N+i-1}) = g(\ldots g(\hat{x}_{k-N}, y_{k-N}), y_{k-N+i-1})$ $i$ times

In order to ensure the convergence of the estimation errors provided by the MHE-PE, the following assumptions must be satisfied

- (A1) $g$ is locally Lipschitz$^1$ with respect to its arguments, with the associated Lipschitz constants $L_\psi^+ \text{ and } L_\psi^-$

- (A2) Denote $\theta x_k \in \theta X$ in the noise-free case and $\theta X$ the estimate of $\theta x_k$. $\forall x_0 \in \hat{X}$, $\forall x_0 \in X$, there exists a class $K$ function $\psi$ such that $\|\theta x_0 - \theta x_k\|^2 \leq \psi(\|\theta x_0 - \theta x_k\|^2)$

- (A3) $g$ satisfies the uniform observability rank condition on $\hat{X}$ with respect to all admissible measurable, i.e. $\exists N > 0$ such that $\forall x_{k-N} \in \hat{X}$, $\forall y \in Y$, rank$(\partial g/\partial x_{k-N}) = n_x$, or equivalently (A3a) $g$ is K-uniformly observable on $\hat{X}$ with

$^1$A function $f(x)$ is said to be locally Lipschitz with respect to its argument $x$ if there exists a positive constant $L_f$ such that $\|f(x') - f(x'')\| \leq L_f \|x' - x''\|$, for all $x', x''$ in a given region of $x$ and $L_f$ is the associated Lipschitz constant.
respect to all admissible measurements, i.e. \( \exists N > 0 \) such that
\[
\forall (\hat{x}', \hat{x}'') \in \hat{X}^2, \forall y \in Y, \text{there exists a K-function } \phi(\cdot) \text{ s.t.}
\]
\[
\phi(||\hat{x}' - \hat{x}'||^2) \leq ||G(\hat{x}', \hat{x}'', y, k_{\text{ref}}) - G(\hat{x}'', \hat{x}'', y, k_{\text{ref}})||^2
\]  
(12)

The observation map \( G(\cdot, \cdot) \) has a finite sensitivity to the estimate, i.e. the K-function \( \phi(\cdot) \) in (12) satisfies:
\[
\delta = \inf_{(\hat{x}', \hat{x}'' \in \mathbb{R}^2, \hat{x}' \neq \hat{x}'')} \frac{\phi(||\hat{x}' - \hat{x}'||^2)}{||\hat{x}' - \hat{x}'||^2} > 0
\]  
(13)

C. Convergence of the estimation errors of the MHE-PE

Theorem 3.1: Denote the estimation error of the MHE at the beginning of the horizon \( k - N \) computed at instant \( k \) by \( e^o_{k-N} = \hat{x}_{k-N} - x_{k-N} \). Suppose that the estimation model described in (8) is identical to the real system and that the system has bounded process and measurement noises. Denote \( \hat{x}_{k,min} \) the smallest eigenvalue of \( R^{-1} \). If \( f \) and \( h \) are locally Lipschitz. Denote \( L_f \) the Lipschitz constant of \( f \), \( 2 \), the pre-estimator \( g \) of the MHE-PE described in (10) is chosen such that assumptions (A1)-(A4) hold and 3. the weight term \( \mu \) of the MHE-PE is selected such that [16]
\[
8\mu(L_f)^2/(\mu + \lambda_{k-min}^2) < 1
\]  
(14)

then the square norm of \( e^o_{k-N} \) is bounded as
\[
\|e^o_{k-N}\|^2 \leq \zeta_{k-N}
\]  
where (a) \( \{\zeta_k\} \) converges exponentially to an asymptotic value \( \zeta_k(\mu) \) and (b) if \( \zeta_k > \zeta_k(\mu) \), then \( \zeta_k > \zeta_k \). The convergence of the estimates of the MHE-PE can be guaranteed by choosing an appropriate value of \( \mu \) satisfying (14) for any value of \( L_f \), once \( \delta \) is calculated using (13).

D. EKF as the pre-estimator

An EKF is chosen as the pre-estimator \( g \), i.e.
\[
\hat{x}_{j+1|k} = g(\hat{x}_{j|k}, y_j) = f(\hat{x}_{j|k}) + K_k(y_j - h(\hat{x}_{j|k}))
\]  
(15)

\[ \forall j \in [k - N, k - 1] \] and where \( K_k \) is the Kalman gain. We assume that there exists a constant \( c_{k,k} \in \mathbb{R} \) such that
\[
c_{k,k} \triangleq \max_{\xi \neq 0} \frac{\|K_k\xi\|}{\|\xi\|} < \infty
\]  
(16)

(16) is verified for the EKF in the MHE-PE since the estimate \( \hat{x} \) is bounded at each instant due to constraints imposed during the optimization. Since \( f \) and \( h \) are locally Lipschitz and (16) holds, \( g \) defined in (15) is locally Lipschitz. (A1) is hence verified. Writing (8) and (15) for a noise-free system and using the locally Lipschitz properties of \( f \) and \( h \) along with (16) show that (A2) is also verified. According to [8], the Lipschitz constant of a function \( \chi \) can be chosen such that \( L^\chi_x = \max \|\delta \chi / \delta x\|_2 \) where the \( \|\cdot\|_2 \) is the operator 2-norm of a matrix. Using MATLAB to plot the values of \( \|\delta f / \delta x\|_2 \) of the simulated trajectories, we find out that \( L^\chi_x = 1.2 \) can be chosen. It is immediate that \( L^\chi_x = 1 \). (A3) is verified using symbolic calculations in MATLAB which implies that (A3a) is verified. Remark that if we find \( \delta \)
\[
\delta \|\hat{x}' - \hat{x}'||^2 \leq \|G_1N_{\hat{x}', \hat{x}'', y, k_{\text{ref}}}) - G_1N_{\hat{x}', \hat{x}'', y, k_{\text{ref}}})||^2
\]
where \( G_1N_{\hat{x}'}, \text{ is the first } n_e \text{ components of } G, \text{ then } (A4) \) is verified. It is verified using symbolic calculation in MATLAB that rank(\( \frac{\partial G_1N}{\partial \hat{z}} \)) = \( n_e \). Hence, the map \( G_1N = \) invertible.

Using the mean-value theorem for multivariable functions [9] and the definition of the matrix lower norm in [18] knowing that \( G_1N = \) invertible, it can be shown that \( \delta = \left\{ \left\| \frac{\partial G_1}{\partial \hat{z}} \right\|^{-1} \right\}^2 \) can be chosen. Using MATLAB to plot the values of \( \left\| \frac{\partial G_1N}{\partial \hat{z}} \right\|^{-1} \) of the simulated trajectories by supposing that the estimate \( \hat{x} \) takes the value of the real state \( x \), we find that \( \delta = 1.6 \times 10^{-15} \). The value \( \mu = 5 \times 10^{-19} \), which allows (14) to be verified, is chosen in this study. To conclude, choosing an EKF as the pre-estimator and this value of weight parameter \( \mu \) can guarantee the convergence of the estimation errors of the MHE-PE.

Note that the difference between the EKF and the MHE-PE with the EKF as pre-estimator is that the EKF is recursive and uses one measurement \( y_k \) at the time while the EKF inside the MHE-PE is initialized at each instant by the optimal state that minimizes the cost function taking into account the sequence of \( N + 1 \) measurements \( \{y_k\}_{k=0}^{N} \).

IV. Numerical Studies

The performances of the MHE-PE with an EKF as pre-estimator, the EKF, the UKF and the RPF will be studied for 3D space debris tracking during atmospheric re-entry via Monte Carlo simulations. Note that in this study, the dynamics of the real system follows (1) which is different from the assumed dynamics in the estimation model (6). Therefore, the robustness of the estimators against model errors will also be studied.

A. Simulations of the real trajectories

The trajectories of 100 Aluminium hollow spherical space debris of width \( w = 3 \) cm are simulated via Monte Carlo simulations using (1). Each debris has different initial altitude \( H_0 \), longitude \( \lambda_{D,0} \), latitude \( \phi_{D,0} \), flight path angle \( \gamma_0 \), flight path angle \( \gamma_0 \) (angle between the local horizontal plane and the velocity of the object), heading angle \( \theta_0 \) (angle between north and the horizontal component of the velocity vector), speed \( ||v_0|| \) and diameter \( D \). These values are generated according to the following uniform distributions: \( H_0 \sim U(69.70,70) \) km, \( \lambda_{D,0} \sim U(2.824443,4.207881) \), \( \phi_{D,0} \sim U(48.820315,48.897227) \), \( \gamma_0 \sim U(-30^\circ,-80^\circ) \), \( \theta_0 \sim U(20^\circ,40^\circ) \), \( ||v_0|| \sim U(5000,100000) \) m/s and \( D \sim U(20,30) \) cm which implies \( m \in U(7.43,18.63) \) kg. The mass \( m \) and the diameter \( D \) are assumed to be constant along the trajectory. Hence, the evolution of the ballistic coefficient depends only on the drag coefficient \( C_D \) which depends only on the Reynolds number \( Re(\gamma, ||v||) \) for a sphere. The expression of \( C_D(\cdot) \) for a sphere can be found in [15].

Using these distributions and equations from section II, A, the trajectories of 100 pieces of debris during 20 s are simulated. The measurements for each trajectory are simulated using (3) by supposing that \( T_s = 0.1 \) s, \( \lambda = 2.336523^\circ \), \( \phi = 48.836080^\circ \), \( \sigma_d = 10 \) m and \( \sigma_el = \sigma_{az} = 0.005^\circ \). These
A. Estimator initialization

For each trajectory, each estimator will be initialized in exactly the same way. $\tilde{r}_0$ and $\tilde{v}_0$ are initialized by two-point differencing method [3] using measurements, i.e., $\tilde{r}_0 = r_m^0 = \left(\begin{array}{c} r_{x,0}^m \\ r_{y,0}^m \\ r_{z,0}^m \end{array}\right)^T$ and $\tilde{v}_0 = \left(\begin{array}{c} \frac{r_{x,0}^m}{T} \\ \frac{r_{y,0}^m}{T} \\ \frac{r_{z,0}^m}{T} \end{array}\right)^T$ where $r_{x,k}^m$, $r_{y,k}^m$, and $r_{z,k}^m$ are the position calculated from the measurements at $k = -1, 0$ using $r_{x,k}^m = \frac{1}{2} \left[ \left(1 - \sin(2\pi m)\right) \theta_0^2 \right]^T$, $r_{y,k}^m = 0$ if $0 < \theta < \frac{\pi}{2}$ or $\theta > \frac{\pi}{2} - \frac{\pi}{2}$, and $r_{z,k}^m = \frac{1}{2} \left[ \left(1 - \sin(2\pi m)\right) \theta_0^2 \right]^T$. It is supposed that the debris is known to be hollow spherical of width $w = 3$ cm and that $H_0 \sim U(69,70)$ km and $||v_0|| \sim U(5000,10000)$ m/s. To study the effect of bad initialization, it is supposed to the estimators that the RPF will be analysed for two different values of the position estimates for $\sqrt{\mathbf{Q}}_1 = 1.5 m/s^2$ and $\sqrt{\mathbf{Q}}_3 = 15 m/s^2$ where $\beta$ has high variation and $\sqrt{\mathbf{Q}}_1 = 1.5 m/s^2$ is adapted for $t \in [10,20]$. We recall that in practice, the real $a$ and $v_0$ are unknown so this quantity cannot be computed. The values of $\tilde{q}$ such that $\sqrt{\mathbf{Q}}_1 = 15$ and $\sqrt{\mathbf{Q}}_3 = 1.5 m/s^2$ are given to the estimators over the entire trajectories to study the robustness of the estimators against poor choices of $\tilde{q}$.

B. Other parameters of the estimators

For the UKF, the tuning parameters $\kappa_{UKF} = 3 - n_x = -6$ is chosen as recommended in [7]. For the RPF, the number of particles $N_p = 2 \cdot 10^4$ is chosen. This choice is made to let the RPF uses the same order of computation time as the MHE-PE. We assume that the RPF does not worth using if it requires more computation time than the MHE-PE but provides less accuracy of the estimates. For the MHE-PE, the constraints on the estimates $\hat{z}$ provided by the MHE-PE are $\tilde{v}_{x,y,z} \in [-0.1,100] km$, $\tilde{v}_{x,y,z} \in [-10^{-4},10^4] m/s$ and $\tilde{a}_{x,y,z} \in [-2500,2500] m/s^2$. The horizon $N = 2n_x = 18$ as recommended in [11] is chosen.

D. Performances of the estimators

The performances of the MHE-PE, the EKF, the UKF and the RPF will be analysed for two different values of the process noise parameter $\tilde{q}$.

1. Non-divergence percentage: In this study, the estimator is said to be non-divergent if its position estimation error at each instant is less than 500 m for all instants $k$. The non-divergence percentages of each estimator are presented in Table I. We remark that the RPF has 0% non-divergence percentage. This may be due to the fact that the PF is sensitive to model errors and bad initialization [12]. The other estimators all provide 100% non-divergence percentages for each case of process noise parameter.

2. Accuracy analysis: The root mean square errors (RMSE) on the norms of the position, the velocity and the acceleration estimates are presented in fig.3. The average root mean square errors (ARMSE), which are the average RMSE over the entire period of the trajectory, on the norm of the position estimates are also presented in Table I. Knowing that a high value of $\tilde{q}$ should be chosen to represent high variation of the real $\beta$ during $t \in [0,10]$ s, we observe that the EKF and the UKF give high estimation errors during this period when $\tilde{q}$ is too small. High errors of the MHE at the beginning of the estimation may due to large number of optimization parameters of the MHE in which local minima can be found.

Hence, we conclude that although the EKF and the UKF provide high non-divergence percentages, they can provide high estimation errors on the position estimates when $\tilde{q}$ is not well chosen. The MHE-PE and the MHE, on the other hand, are shown to be robust against a bad choice of $\tilde{q}$. 

Table I: Non-divergence percentage and ARMSE on the norm of the position estimates for $\sqrt{\mathbf{Q}}_1 = 1.5 m/s^2$ and $\sqrt{\mathbf{Q}}_3 = 15 m/s^2$

<table>
<thead>
<tr>
<th>Estimator</th>
<th>$\sqrt{\mathbf{Q}}_1 = 1.5 m/s^2$</th>
<th>$\sqrt{\mathbf{Q}}_3 = 15 m/s^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Non-div. %</td>
<td>ARMSE [m]</td>
</tr>
<tr>
<td>EKF</td>
<td>100</td>
<td>19.04</td>
</tr>
<tr>
<td>UKF</td>
<td>100</td>
<td>15.76</td>
</tr>
<tr>
<td>RPF</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>MHE-PE</td>
<td>100</td>
<td>8.29</td>
</tr>
<tr>
<td>MHE</td>
<td>100</td>
<td>11.10</td>
</tr>
</tbody>
</table>

Fig. 2: Evolution of the ballistic coefficient of one of the simulated trajectories (left) and distribution of $|a_0|$ in our studies (right).
to the EKF, the UKF, the RPF, the MHE-PE and the MHE to have the trajectory estimates. The MHE-PE and the MHE are shown to be robust against poor choice of process noise parameter while the EKF, the UKF and the RPF are not. The MHE exhibits high errors at the beginning of the estimation which may due to presence of local minima. The MHE-PE is shown to require ~5 times less computation time than the MHE since it does not require the estimation of the process noise over its horizon, which reduces the number of optimization parameters. The computation time of the MHE-PE in this study is not sufficiently small for a real-time space debris tracking yet. Still, it can still be reduced when the MHE-PE is combined to fast optimization techniques or programmed in a faster language.

**REFERENCES**


