Output invisible control allocation with steady-state input optimization for weakly redundant plants.

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Abstract—In this paper, the problem of control allocation is revisited focusing on weakly redundant plants, that is plants having more inputs than outputs but such that no input can be equivalently substituted by a combination of the others. Considering the case of asymptotically constant references, it is shown that input allocation can be used to optimize the steady state input, without altering (even during transients) the output response induced by any a priori given controller. Moreover, a general structure for the allocator is proposed, consisting of a series connection of a steady-state optimizer and an annihilator.

I. INTRODUCTION

Control allocation is an important problem in several applications where multiple actuators are present, and their number is larger than the number of outputs to be controlled; several such examples arise in the context of aircraft control, marine vehicles, double actuator positioning in hard disks, and several others (see the discussions in [1], [2], [3], [4], [5], [6], [7] and references therein). In particular, this paper presents some extensions of the allocation approach proposed in [7], exploited in a sequence of papers (especially in the context of control for plasma confinement in nuclear fusion reactions in Tokamaks [8], [9] and automotive applications [10], [11]) and recently considered in the regulation context [12], [13].

In [7], linear plants (described as \( \dot{x} = Ax + Bu, y = Cx + Du \)) are classified as strongly redundant if the columns of matrix \([B', D']\) are not all linearly independent (which is equivalent to say that at least one of the inputs can always be exactly replaced by a suitable combination of the remaining inputs), and weakly redundant otherwise; then, in order to optimize the steady-state input (with respect to some cost function), [7] considers two different allocation problems for the two cases, in particular for strongly redundant plants allocation is performed without perturbing in any way the output (and state) response of the plant (which is achieved by generating allocation signals in the null space of \([B', D']\)), whereas for weakly redundant plants the more modest objective of not perturbing the plant output response only at the (constant) steady-state is considered.

The main novelty considered in this paper relates to considering situations where the allocator is still required to produce completely output-invisible corrections to the plant input in order to achieve a reduction of the steady state input (with respect to some suitable norm or cost function), but the plant is only assumed to be weakly redundant; in other terms, the distinction between output-invisible versus steady-state output-invisible allocation is made, and the first, stronger requirement is shown to be achievable even when the plant is not strongly redundant. Some work closely related to the developments in this paper can be found in [13], the main difference being that the geometric approach provided in [13] is replaced here by a complementary, essentially algebraic approach more modestly aimed at direct computation of the compensator, as well as some consideration on the nonlinear extension of the proposed results. Relevant discussion about steady-state output-invisible allocation can be found in the companion paper [14].

It is also shown in this paper that the allocator can be given a suggestive structure, namely the cascade of two (in a suitable sense) subsystems, the first one called the steady-state optimizer, being either a static map or a dynamic system implementing a gradient-like algorithm, in charge of determining the optimal steady-state allocation, and the second one called the annihilator in charge of modulating the optimal input correction in such a way to guarantee the invisibility of such correction from the plant output. Such an architecture is in essence already implemented in [7] and following papers, but never made explicit, with the consequence that the ensuing modularity (allowing, for example, to substitute at will a dynamic gradient optimizer with a static optimizing map) is not exploited in those references.

One application of the developed theory is in the case of allocation in the presence of several actuators characterized by different non trivial dynamics, considered in the past e.g. in [15], [16], and more recently for an automotive application in [10], [11], where a hybrid vehicle is considered and input allocation is exploited in order to optimally combine the torques generated by a combustion engine and an electric motor. Compared with the approach in those papers, the present design allows a systematic design and a radically simpler stability analysis.

Finally, some initial contribution towards the extension of the same results in a nonlinear context are proposed. While a more detailed treatment of the nonlinear case is the subject
of current investigation and will be reported in future work, we feel that the proposed framework (geared toward the nonlinear case) is already an interesting contribution.

Novel contribution. Summarizing the previous discussion, the novelties proposed in this paper consist in:

- clearly distinguishing the requirement of output invisible allocation from the property of the system of being strongly or weakly redundant;
- completely solving the output invisible allocation problem for weakly redundant plants, at least in the linear case;
- proposing a general, flexible architecture for the allocator as the cascade of an optimizer and an annihilator;
- proposing a systematic approach to deal with actuators having different dynamics.

II. Problem statement

Consider a nonlinear system $\mathcal{P}$ described by

$$\begin{align*}
\dot{x} &= f(x,u) \\
y &= h(x) 
\end{align*}$$

with $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ and $y \in \mathbb{R}^p$. An a priori given nonlinear controller $\mathcal{C}$ for plant $\mathcal{P}$ in (1) is supposed to be available, described by the equations

$$\begin{align*}
\dot{x}_c &= f_c(x_c,u_c,r) \\
y_c &= h_c(x_c,u_c,r)
\end{align*}$$

where $r$ is a measured exogenous input (e.g. a reference signal). The interconnection equations between (1) and (2) are given by:

$$\begin{align*}
u_c &= y_c, \\
u &= y_c + w,
\end{align*}$$

where $w$ is an additional input that will be exploited for input allocation. In order for the theory to be developed to apply, controller (2) can be quite arbitrary, as long as it satisfies the following assumption, where $\Sigma$ denotes the closed-loop system (1), (2), (3).

For the following developments, the notions of input-to-state-stability (ISS) and asymptotically-constant-input asymptotically-constant-state (ACIACS) will be useful. For simplicity, such properties will be assumed to hold globally, although suitable local versions (with restrictions on the relevant classes of signals and states) can be used in specific contexts, if needed. In particular, a system will be said to be ACIACS if its state response asymptotically converges to a constant $x_\infty$ every time its input asymptotically converges to a constant $u_\infty$, with $x_\infty$ uniquely determined by $u_\infty$; so, there exists an input to state static map associating each value of $u_\infty$ to a unique value $x_\infty$ of the state according to the relation $x_\infty = \Omega^x(u_\infty)$, and an input to output static map associating each value of $u_\infty$ to a unique value $y_\infty$ of the output according to the relation $y_\infty = \Omega^y(u_\infty)$. On the other hand, the classic definition of ISS (see e.g. [17]) requires that

$$|x(t)| \leq \beta(|x(0)|,t) + \gamma(\sup_{\tau \in [0,t]} |u(\tau)|), \quad \forall t \geq 0,$$

where $\gamma(\cdot)$ is a $K$ function (that is, $\gamma(0) = 0$, and it is non-negative and strictly increasing) and $\beta(\cdot,\cdot)$ is a $KL$ function (that is, it is a $K$ function in the first argument, whereas in its second argument is decreasing and asymptotically tends to zero when the second argument tends to infinity). It is well known that the cascade connection of two ISS systems is ISS itself.

**Assumption 1:** The closed-loop system $\Sigma$ is ISS and ACIACS with respect to $(w,r)$.

**Remark 1:** At least when (1), (2) are actually linear, Assumption 1 is satisfied as soon as the closed-loop system $\Sigma$ is asymptotically stable. In general, ACIACS further implies that when the signals $(w,r)$ converge to constant values ($w_\infty, r_\infty$) then also the output of each subsystem in the closed-loop system converges to a unique constant value (depending on ($w_\infty, r_\infty$)).

The problem of interest in this paper (compare Fig. 1 and Fig. 2) consists in designing a device, called (input) allocator, described by the equations

$$\begin{align*}
\dot{x}_a &= f_a(x_a,u_a) \\
y_a &= h_a(x_a,u_a)
\end{align*}$$

which, suitably connected to the closed-loop system $\Sigma$ according to

$$\begin{align*}
u_a &= \begin{bmatrix} y_c \\ x \\ r \end{bmatrix}, \\
w &= y_a,
\end{align*}$$

Fig. 1: The original closed loop system $\Sigma$ (without input allocation).

Fig. 2: The input allocated closed loop system $\Sigma_a$. The dashed arrows denote inputs to the allocator used in special cases.
exploits the input redundancy in order to optimize the steady-state value of the plant input.

Remark 2: In many cases of interest, the allocator can be simply fed measurements of $y_c$, especially when plant (1) is actually linear. Measurements of $x$ (or possibly $y$) can be useful (or needed) in the nonlinear case in order to guarantee output invisible allocation. Measurements of $r$ can be useful in order e.g. to bound the norm of the allocation signal proportionally with respect to the norm of the reference signal; details on this additional variation will be reported in a forthcoming paper.

It is also worth mentioning that, while the dynamics (4) are chosen as a classic continuous-time nonlinear dynamic system, the possibility of using an hybrid system as allocator can be considered as well, and opens very interesting scenarios, especially considering the modularity of the allocator architecture that will be proposed in Section III.

Considering the setup in Fig. 2, the formal definition of the problem of interest is the following, where

- $\Sigma_a$ denotes the closed-loop system (1), (2), (3), (4), (5);
- $W_a$ is the input-output response from $r$ to $y$ of $\Sigma$, when $w \equiv 0$;
- $W_a$ is the input-output response from $r$ to $y$ of $\Sigma_a$;
- $J(\cdot)$ is a strictly convex cost function defined on $\mathbb{R}^m$;
- $\Sigma_y$, $\Sigma_c$ are the steady-state values of $y$ and $y_c$ in $\Sigma$ corresponding to $r_\infty$.

\textbf{Problem 1:} Design an input allocator (4) such that the allocated closed-loop system $\Sigma_a$

\begin{itemize}
  \item i) is ISS and ACIACS with respect to $r$;
  \item ii) $W_a$ and $W_a$ (from the same initial states of (1) and (2)) coincide;
  \item iii) the steady-state plant input $u_\infty = y_c + y_{a,\infty}$ satisfies
  \[ J(u_\infty) = \min_{s \in U(y_{c,\infty})} J(s), \]
\end{itemize}

where $U_\Sigma(y_{c,\infty})$ is the set of all constant inputs to (1) as a subsystem of $\Sigma$ which are compatible with the constant output $y_{c,\infty}$.

Remark 3: In item iii) in Problem 1, the value $y_{c,\infty}$ appearing in $u_\infty = y_c + y_{a,\infty}$ is exactly the same value that would appear in $\Sigma$ for $w \equiv 0$ and the same reference $r$. In fact, by item ii) in Problem 1, the output $y$ of (1) is not modified at all by the presence of the allocator (hence the name output invisible) in $\Sigma_a$ with respect to its value in $\Sigma$, so that the output $y_c$ of (2) is unmodified as well, and then \textit{a fortiori} the steady-state value of $y_c$, namely $y_{c,\infty}$, is the same in $\Sigma$ and $\Sigma_a$.

III. PROPOSED ARCHITECTURE

The solution to Problem 1 proposed in this paper hinges upon an architecture, shown in Fig. 2 and Fig. 3, which extends the allocator structure in considered in [7]; for a detailed comparison, see Remark 4.

The architecture in Fig. 2 considers an essentially parallel allocator, that is the allocator is placed in parallel to the standard interconnection between the controller (2) and the plant (1); the connection of a steady-state optimizer and an annihilator.

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\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{architectural_diagram}
\caption{The internal structure of the allocator: a cascade connection of a steady-state optimizer and an annihilator.}
\end{figure}
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Fig. 3: The internal structure of the allocator: a cascade connection of a steady-state optimizer and an annihilator.

As for the steady-state optimizer, consider the interconnection in Fig. 4, and assume that $r$ converges to a constant value $r_\infty$, so that also $y_c$ and $y$ converge to constant values $y_{c,\infty}$ and $y_{\infty}$ by ACIACS of $\Sigma$, cfr Assumption 1). Then a steady-state optimizer is a (static or dynamic) ISS and ACIACS system such that, given as input the output $y_c$ of

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\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{optimizer_diagram}
\caption{The open-loop characterization of the steady-state optimizer; signal $r$ is an optional input to $Opt$.}
\end{figure}
```

Global invertibility allows to design a gradient based optimizer (as described in Section IV-A, but considering the optimization directly in the argument of $J(u)$, that is $u$) and invert the static map of the annihilator in order to obtain a unique value for the optimizer output.

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\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{annihilator_diagram}
\caption{The (quasi) open-loop characterization of the annihilator; the feedback signal $x$ is used in the nonlinear case.}
\end{figure}
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Fig. 5: The (quasi) open-loop characterization of the annihilator; the feedback signal $x$ is used in the nonlinear case.
the controller (2) (as a subsystem of Σ) and possibly r, it
provides an output which asymptotically converges to a value
v∞ such that u∞ − y∞ = Ωopt An(v∞), where u∞ is defined in
item iii) of Problem 1, namely is the input to (1) minimizing
the cost J (note that in case several).

Some details on the construction of suitable annihilators
and optimizers will be given in the following sections, both
for the linear and the nonlinear cases. Based on the above
premises, it is easy to show that any allocator with the
structure in Fig. 3 solves Problem 1, provided that the
optimizer and the annihilator are ISS and ACIACS. The
formal statement is the following.

**Theorem 1:** Under Assumption 1, Problem 1 is solved by
the input allocator (4) with the architecture in Fig. 3 with
ISS and ACIACS annihilator Ann and optimizer Opt.

**Proof:** The proof of Theorem 1 closely parallels the
proof of [7, Theorem 1], and then it is only sketched. The key
point (proved later) consists in noting that when the structure
in Fig. 2 and Fig. 3 is used, the values taken by signal v are
independent from the presence of the annihilator Ann, so
item i) is proved, that is Σa is ISS and ACIACS, if and
only if the same properties hold for the cascade connections
in Fig. 4 and Fig. 5, which are easily proven to be ISS
and ACIACS by standard results on cascade systems, as
soon as Σ, Opt and Ann have the same properties. The fact
that also item iii) of Problem 1 is satisfied is then a
trivial consequence of the structure in Fig. 2 and Fig. 3, the
ACIACS property and the definition of Opt (together with
the global invertibility of the static map of the annihilator,
see footnote 1).

In order to prove the above key point, note that the responses
of controller C and optimizer Opt in Fig. 2 coincide with the
responses of C and Opt in Fig. 4, since the signal yd of the
allocator in Σa is actually produced by an annihilator (cfr
Fig. 3), and then the output y of plant P in Fig. 2 coincides
with the corresponding output of P in Fig. 4 (which also
implies that item ii) in Problem 1 is satisfied).

**Remark 4:** It is worth mentioning that while a feedback
(instead of parallel) architecture for the allocator was pro-
posed in [7], subsequent papers explicitly adopted a parallel
structure. On the other hand, the decomposition of the internal
structure of the allocator as a cascade of an optimizer
and an annihilator was never really made explicit in those
papers, and it is believed that this fact contributed to the
point of view used in [7], that output-invisible allocation
is necessarily linked to strong input redundancy, that is the
presence (in the linear case) of a non-trivial ker([B' D']
(note that in case several).

In this respect, there is quite a strong connection with the
results in [13], which characterizes (from a geometric point
of view) the structure of a system which admits motions of the
kind exploited by our annihilator (that is, invisible from the
plant output), without assuming that rank ([B' D'] < m,
as was the case in [7]. On the other hand, the developments in
[13] are focused on the classic problem of output regulation
[18], [19], (that is, there is an underlying objective of zeroing
an error output despite the presence of exogenous signals
generated by a known Poisson stable exosystem), whereas
such problem is completely out of scope here.

In order to have more constructive recipes (at least for
specific classes of plants), the case of linear systems is
considered next.

### IV. LINEAR SYSTEMS

Let the state-space description of (1) be

\[ \dot{x} = Ax + Bu, \]
\[ y = Cx, \]

and (2) be given by

\[ \dot{x}_c = A_c x_c + B_c u_c, \]
\[ y_c = C_c x_c. \]

Recall that for linear systems the properties of global
exponential stability, ACIACS, ISS are all equivalent to local
asymptotic stability of the origin, so that Assumption 1 can
be immediately simplified as follows.

**Assumption 2:** The closed-loop system Σ is asymptoti-
cally stable.

In order to avoid trivialities, the following is explicitly
assumed.

**Assumption 3:** rank (B) = m and rank ([A−sI B]) =
\[ n + p \] for s = 0.

**A. Optimizer design**

Considering an approach closer in spirit to some consider-
ations in [7], [13], [12], a gradient rule is defined in order
to minimize a convex cost function depending of the plant
input. Let \( W_A (s) \) be the transfer matrix of a linear time
invariant annihilator (whose design will be discussed in the
following two subsections), and let \( \Omega_{Ann}, C_{opt} \) be such
that for the static gain \( W_A (0) \) it holds that \( W_A (0) C_{opt} = \Omega_{Ann} \),
with \( \Omega_{Ann} \) having full column rank equal to the rank of
\( W_A (0) \); clearly, if \( W_A (0) \) is already full column rank,
then \( \Omega_{Ann} = W_A (0) \) and \( C_{opt} = I \). Define the optimizer
dynamics:

\[ \dot{x}_{opt} = -\alpha \nabla J(y_c + \Omega_{Ann} x_{opt}), \quad \alpha > 0, \]
\[ v = C_{opt} x_{opt}, \]

where \( \alpha > 0 \) is an arbitrary gain which can be used to
increase the convergence rate. Using the above notations,
the following proposition holds.

**Proposition 1:** The optimizer (9) is ISS, ACIACS and
converges to the minimum of J, for any \( \alpha > 0 \).
Proof: Since the composition of a convex function with an affine function is still convex, standard results show that the gradient dynamics (9a) converges to a local minimum, which by convexity is also a global minimum. By the full column rank property of $\Omega_{An}$, such minimum is also unique. Since $\Omega_{An}$ and $W_{An}(0)$ have the same image, and $W_{An}(s)$ is the transfer matrix of an annihilator, it follows that the proposed optimizer $Opt$ in series with the annihilator with transfer matrix $W_{An}(s)$ does not alter the $y$ response of plant $P$. Finally, it is easy to see that when $y_0$ converges to a constant value, the overall plant input obtained by implementing (9) converges to the corresponding optimal input.

The reason why the optimizer design has been described before the annihilator design is to stress that these two designs are completely independent, except for the fact that the annihilator static gain $W_{An}(0)$ is needed in order to compute $C_{opt}$ and $\Omega_{An}$, so that in practice the annihilator must be designed before the optimizer design is completed, in order to substitute suitable values in (9). Apart from this fact, however, the allocator design is fully modular: an annihilator can be replaced by any other annihilator (provided that the gain in the optimizer is suitably updated, if different) and conversely any optimizer can be replaced by a different optimizer, perhaps static instead of dynamic, or optimizing a completely different convex cost; and such changes can be done even during operation, without affecting the plant output (the allocator remains output-invisible).

Remark 5: In the special case when $J$ is a quadratic function, the optimizer (9) is a linear time invariant system, which might be a desirable feature in many applications.

The optimizer block can be designed in several ways, not necessarily based on the gradient dynamics (9). One possible alternative approach consists in explicitly designing a static map providing the optimal input (see e.g. [20]); such an approach is applicable in some special cases, some of which particularly relevant in terms of applications, e.g. the minimization of the $L_{\infty}$ norm of the steady-state control input [20], [21].

Note that only the constant steady-state maps are relevant in order to design the proposed optimizer. Moreover, such maps can be suitably saturated (or, more generally, modified) in such a way to guarantee e.g. better transients with smaller overshoots, without impairing the stability properties of the overall allocated closed-loop system $\Sigma_a$ due to the output-invisible nature of the proposed allocator (see the proof of Theorem 1).

B. Annihilator design: a polynomial factorization approach

The simplest possible approach to design an annihilator can be based on the transfer matrix of (7) and its left coprime polynomial factorization [22, Ch. 7]

$$W(s) = C(sI - A)^{-1}B = D^{-1}(s)N(s),$$

where $D(s)$, $N(s)$ are polynomial matrices (with $D(s)$ square and nonsingular as a polynomial matrix, namely having a determinant different from the zero polynomial).

By exploiting standard results based on elementary row and column operations, the Smith form $S_N(s)$ of matrix $N(s)$ can be obtained by using elementary operation, that is by pre- and post-multiplication of $N(s)$ by unimodular polynomial matrices $^2 L(s)$ and $R(s)$. Since rank $(W) = p$ as a rational matrix due to Assumption 3, the Smith form $S_N(s)$ of $N(s)$ will be given by

$$S_N(s) = \begin{bmatrix} S_N^0(s) & 0_{p \times (m-p)} \end{bmatrix} = L(s)N(s)R(s),$$

with $S_N^0(s)$ square, diagonal and full rank. It is then immediate to see that partitioning $R(s) = [R_1(s) \ R_2(s)]$ with $R_2(s)$ having $(m-p)$ columns, it holds that

$$W(s)R_2(s) = D^{-1}(s)N(s)R_2(s) = 0.$$ 

Hence, a stable annihilator can be obtained from $R_2(s)$ by simply selecting a polynomial invertible matrix $\Psi(s)$ such that the transfer matrix

$$W_{An}(s) = R_2(s)\Psi^{-1}(s),$$ (10)

is proper and stable. An explicit algorithm is as follows.

Algorithm 1: Annihilator design I.

- Compute a left coprime polynomial factorization $D^{-1}(s)N(s)$ of the transfer matrix $W(s)$ of (7).
- Compute unimodular polynomial matrices $L(s)$ and $R(s)$ such that $L(s)N(s)R(s)$ is in Smith form.
- Define $R_2(s) = R(s)\begin{bmatrix} I_{m-p} \end{bmatrix}$.
- Choose $\Psi(s) = \text{diag}(\psi_1(s), \ldots, \psi_{m-p}(s))$ such that each $\psi_k(s)$ has real coefficient, degree not smaller than the highest degree in the $h$-th column of $R_2(s)$, and all roots in a desired set $\mathcal{C}_q \subset \{s : \text{Re}(s) < 0\}$.
- Define the annihilator for (7) as any minimal realization of $W_{An}(s) = R_2(s)\Psi^{-1}(s)$.

The previous discussion immediately yields next result, stating the effectiveness of Algorithm 1.

Proposition 2: Algorithm 1 provides an annihilator for (7) satisfying the requirements in Theorem 1.

Since $\mathcal{C}_q$ can be chosen arbitrarily far to the left of the imaginary axis in the complex plane, the following corollary is immediate.

Corollary 1: An annihilator with arbitrarily fast dynamics can be designed under Assumption 3.

Corollary 1, coupled with the fact that the optimizer can be chosen static or anyway with arbitrarily fast dynamics, shows that arbitrarily fast output invisible allocation is possible even in weakly redundant plants.

C. Annihilator design: a geometric design approach

In the linear case, the result in the previous section is very simple and effective in order to deal with the allocation problem by using the proposed architecture. However, it is not trivial to directly extend such an approach to a nonlinear context, which is the subject of the next section. For this...
reason, a similar result is now derived by using a different, geometric route, more amenable to a nonlinear extension (at least for the case of minimum phase, input affine nonlinear systems having a well defined vector relative degree). A more in depth treatment of very similar ideas, although with a somewhat different objective, can be found in [13].

Recall that, for (7), the symbol \( \mathcal{V}^* \subset \mathbb{R}^n \) and \( \mathcal{R}^* \subset \mathcal{V}^* \) denote, respectively, the largest controlled-invariant subspace and the largest controllability subspace contained in \( \ker C \) [23], i.e., \( \mathcal{V}^* \) is the set of initial conditions for which there exists an input function such that the ensuing output is identically zero, and \( \mathcal{R}^* \) is the controllable subset (actually, a subspace) of \( \mathcal{V}^* \), namely the set of initial conditions for which there exists an input function able to steer the state to zero in finite time while keeping the output identically zero. It is well known [24] that \( \mathcal{V}^* \) is the largest subspace \( \mathcal{V} \subset \mathbb{R}^n \) such that

\[
\begin{bmatrix} A \\ C \end{bmatrix} \mathcal{V} \subset (\mathcal{V} \times 0) + \text{Im} \begin{bmatrix} B \\ 0 \end{bmatrix},
\]

(11)
or equivalently the largest subspace \( \mathcal{V} \subset \mathbb{R}^n \) such that there exists \( F \in \mathbb{R}^{m \times n} \) ensuring

\[
(A + BF)\mathcal{V} \subset \mathcal{V}, \quad CV = 0,
\]

(12)
and such a matrix \( F \) satisfying (12) is called a friend of \( \mathcal{V} \). Obviously, \( \mathcal{R}^* \subset \mathcal{V}^* \); moreover, any friend of \( \mathcal{V}^* \) is also a friend of \( \mathcal{R}^* \) [24, Th. 7.14]. On the other hand, if \( \mathcal{R}^* \neq \mathcal{V}^* \) there exist infinitely many friends of \( \mathcal{R}^* \) that are not friends of \( \mathcal{V}^* \) (this fact is used in the sequel). Let \( \rho^* \) be the dimension of \( \mathcal{R}^* \) and \( \nu^* \) be the dimension of \( \mathcal{V}^* \). It is worth recalling that the set of invariant zeros of (7) coincides with the spectrum of the map induced in \( \mathcal{V}^*/\mathcal{R}^* \) by \( AF := A + BF \), where \( F \) is a generic friend of \( \mathcal{V}^* \). Even if \( M = \mathbb{R}^u \times \mathbb{R}^\nu \) is a not invertible (or not even square) matrix, it is customary [25] to define the subspace \( M^{-1}\mathcal{V} := \{v \in \mathcal{V} : Mv \in \mathcal{V}\} \), for \( \mathcal{V} \) a subspace of \( \mathbb{R}^u \). It is worth recalling that software routines for the computation of all the mentioned subspaces and matrices are readily available. 3

Following an approach similar to the one used in [12], the following algorithm can be devised.

**Algorithm 2: Annihilator design II.**

1. Define invertible matrices \( T \in \mathbb{R}^{n \times n} \) and \( N \in \mathbb{R}^{m \times n} \) where \( T^{-1} = \begin{bmatrix} T_1 & T_2 \end{bmatrix} \) with \( \text{Im} (T_1) = \mathcal{R}^* \) (and \( T_2 \) is such that \( \text{det}(T_1 T_2) \neq 0 \)), and \( N^{-1} = \begin{bmatrix} N_1 & N_2 \end{bmatrix} \) with \( \text{Im} (N_1) = B^{-1} \mathcal{R}^* \) (and \( N_2 \) is such that \( \text{det}(N_1 N_2) \neq 0 \)).

2. Define new coordinates \( \bar{x} = Tx, \bar{u} = [\bar{u}_1 \bar{u}_2]' = Nu \) yielding the following block structure:

\[
\begin{align*}
\bar{A} &= TAT^{-1} = \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} \\ \bar{A}_{21} & \bar{A}_{22} \end{bmatrix}, \\
\bar{B} &= TB = \begin{bmatrix} \bar{B}_{11} & \bar{B}_{12} \\ 0 & \bar{B}_{22} \end{bmatrix}, \\
\bar{C} &= CT^{-1} = \begin{bmatrix} 0 & \bar{C}_2 \end{bmatrix},
\end{align*}
\]

(13a)

(13b)

(13c)

3E.g. in the freely available online Geometric Control Toolbox.

- Compute a friend \( \bar{F} = \begin{bmatrix} \bar{F}_{11} & 0 \\ \bar{F}_{21} & \bar{F}_{22} \end{bmatrix} \) of \( \mathcal{R}^* \) such that the spectra of \( \bar{A}_{11} := (\bar{A}_{11} + \bar{B}_{12} \bar{F}_{21} + \bar{B}_{11} \bar{F}_{11}) \) and \( \bar{A}_{22} := (\bar{A}_{22} + \bar{B}_{22} \bar{F}_{22}) \) both lies in \( \mathbb{C}_g \).

- Define the annihilator as:

\[
\begin{align*}
\dot{x}_A &= \bar{A}_{11} \bar{x}_A + \bar{B}_{11} \bar{u}_1, \quad (14a) \\
y_A &= N^{-1} \left( \begin{bmatrix} \bar{F}_{11} \\ \bar{F}_{21} \end{bmatrix} \bar{x}_A + \begin{bmatrix} I_{m-p} \\ 0 \end{bmatrix} \bar{u}_1 \right), \quad (14b)
\end{align*}
\]

with \( x_A(0) = 0 \).

Note that the choice of \( \bar{F} \) as indicated is always possible since the plant is necessarily stabilizable (in view of Assumption 2), and then \( \mathcal{R}^* \) is both externally stabilizable (by [25, Property 4.1.16]) and internally stabilizable (by definition of \( \mathcal{R}^* \)). Since (14) is clearly stable, in order for (14) to be a suitable annihilator for the proposed allocator architecture, it is necessary just to show that for all \( \bar{u}_1 \) the forced output response of (7) to the input \( u = y_A + y_c \) coincides with the response to the input \( u = y_c \), which by linearity is tantamount to show that the forced output response of (7) to the input \( u = y_A \) is zero. Hence, the following result holds.

**Proposition 3:** Algorithm 2 provides an annihilator for (7) satisfying the requirements in Theorem 1.

**Proof:** Consider the system (7) in the coordinates \( \bar{x}, \bar{u} \), and assume \( \bar{x}(0) = 0 \) (since a forced response must be computed) and \( \bar{u} = \bar{y}_A \) where

\[
\bar{y}_A := N y_A = \begin{bmatrix} \bar{F}_{11} \\ \bar{F}_{21} \end{bmatrix} \bar{x}_A + \begin{bmatrix} I_{m-p} \\ 0 \end{bmatrix} \bar{u}_1.
\]

Letting the state, input and output variables of (7) be decomposed according to (13) as

\[
\begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix}, \quad \begin{bmatrix} \bar{u}_1 \\ \bar{u}_2 \end{bmatrix}, \quad \begin{bmatrix} y \end{bmatrix} = C_2 \bar{x}_2(t),
\]

the claim will be proven by showing that the subspace \( \bar{x}_1 = x_A, \bar{x}_2 = 0 \), on which \( y = C_2 \bar{x}_2 \) is zero, is an invariant for all choices of \( \bar{u}_1 \). The overall dynamics of the cascade of (14) followed by (7) is given by

\[
\begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} + \begin{bmatrix} B_{11} \\ B_{22} \end{bmatrix} \begin{bmatrix} \bar{F}_{11} \\ \bar{F}_{21} \end{bmatrix} \begin{bmatrix} \bar{x}_A \\ \bar{x}_A \end{bmatrix} + \begin{bmatrix} 0 \\ \bar{B}_{22} \end{bmatrix} \bar{u}_1.
\]

(15)

Substituting in this equation the relation (expressing \( \bar{x}_1 = x_A, \bar{x}_2 = 0 \)):

\[
\begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \bar{x}_A \end{bmatrix} = \begin{bmatrix} I \\ 0 \\ I \end{bmatrix} \chi,
\]

where \( \chi \) is an arbitrary vector (of suitable dimension), it follows that

\[
\begin{bmatrix} \dot{\bar{x}}_1 \\ \dot{\bar{x}}_2 \\ \dot{\bar{x}}_A \end{bmatrix} = \bar{A}_{11} \begin{bmatrix} I \\ 0 \\ I \end{bmatrix} \chi + \begin{bmatrix} \bar{B}_{11} \\ 0 \\ \bar{B}_{11} \end{bmatrix} \bar{u}_1,
\]

so that the subspace \( \bar{x}_1 = x_A, \bar{x}_2 = 0 \), is invariant.
The transfer matrix of (14) can be computed as
\[
W_{An}(s) = N^{-1} \bar{W}_{An}(s),
\]
and clearly its realization (14) satisfies the ISS and ACIACS properties (which for a linear system correspond to asymptotic stability of its equilibrium at the origin) required for the proposed allocation approach. Note also that Corollary 1 applies here too (since arbitrarily fast dynamics can be assigned to \( \bar{A}_{11} \)).

V. ACTUATOR DYNAMICS

The aim of this section is to suggest two ways by which the previously proposed approach can be brought to bear on the case where nontrivial actuator dynamics are present; this discussion is mainly motivated by the similar extension provided in [11], [10] of the results in [7].

First, the actuator dynamics can be included in the plant dynamics (simply by computing the cascade of the actuator and the plant transfer matrices), and then the proposed method can be directly applied. Note that in order to penalize in the cost function the actual plant input (and not just the actuator command) it is enough to replace \( J(u) \) by \( J(G(0)u) \), where \( G(s) \) is the actuator transfer matrix.

An alternative approach is based on an idea in [11], [10], and might be attractive when the same plant is supposed to work with different sets of actuators (e.g. like a hybrid car which is sold with different combinations of combustion engines and electric motors). The approach is based on computing a single annihilator tailored to the plant, and then introducing suitable filtering actions in order to allow the annihilator and the actuator transfer matrix commute.

In this second case, let \( G(s) \) denote the stable diagonal transfer matrix associated to the actuator dynamics. It is easily seen that it is always possible to find a stable transfer function \( \kappa(s) \) with \( \kappa(0) \neq 0 \) and a proper stable diagonal transfer matrix \( G^-(s) \) such that
\[
G(s)G^-(s) = G^-(s)G(s) = \kappa(s)I. \tag{17}
\]

Let \( W_{An}(s) \) be the transfer matrix of an annihilator designed for plant \( P \) in (7), so that \( W(s)W_{An}(s) = 0 \); it is then easily seen that
\[
W(s)G(s)G^-(s)W_{An}(s) = W(s)\kappa(s)W_{An}(s) = \kappa(s)W(s)W_{An}(s) = 0,
\]
so that \( \bar{W}_{An}(s) = G^-(s)W_{An}(s) \) is an annihilator for the actuator-plant cascade. The only additional care to be taken in designing an associated steady-state input optimizer consists in using \( \bar{W}_{An}(0) \) in place of \( W_{An}(0) \) and in replacing \( J(y_c + W_{An}(0)v) \) with \( J(G(0)(y_c + W_{An}(0)v)) \), taking into account that the actual plant input is the output of the actuators.

VI. NUMERICAL TEST

In this section, the geometric design approach proposed in Section IV-C, with the modification proposed in Section V in order to deal with actuator dynamics, is applied on a linear system. Let us consider the following plant:
\[
\bar{A} = \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} \\ \bar{A}_{21} & \bar{A}_{22} \end{bmatrix} = \begin{bmatrix} -1 & 0 & -1 & 1 \\ -2 & -4 & -1 & -1 \\ 1 & 1 & 0 & -1 \\ -2 & -1 & -1 & -3 \end{bmatrix},
\]
\[
B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ -1 & 2 & -1 \\ 2 & 0 & 2 \end{bmatrix},
\]
\[
C = [0 \ 0 \ 0 \ 1].
\]

The actuators dynamics is given by \( u(s) = G(s)v(s) \) with
\[
\dot{u}_1 = -0.5u_1 + v_1,
\]
\[
\dot{u}_2 = -2u_2 + v_2,
\]
\[
\dot{u}_3 = -u_3 + v_3,
\]
\[
G(s) = \begin{bmatrix} \frac{2}{1+s} & 0 & 0 \\ 0 & \frac{1}{1+s} & 0 \\ 0 & 0 & \frac{1}{1+s} \end{bmatrix}.
\]

The identity \( G(s)G^-(s) = G^-(s)G(s) = \kappa(s)I \) holds with \( \kappa(s) = \frac{1}{2 + 3s + s^2} \) if \( G^-(s) \) is defined as
\[
G^-(s) = \begin{bmatrix} \frac{1+2s}{2+6s+2s^2} & 0 & 0 \\ 0 & \frac{1+s}{1+s} & 0 \\ 0 & 0 & \frac{1+s}{1+s} \end{bmatrix},
\]

The following dynamic controller has been considered
\[
\dot{x}_c = -x_c + B_c(y_p + r(t)),
\]
\[
y_c = x_c,
\]

with \( B_c = [0 \ 0 \ 1]' \) and reference signal
\[
r(t) = \begin{cases} 3 & t \in (0, 35) \\ -1 & t \in [35, 70) \\ 1.5 & t \geq 70 \end{cases}.
\]

By the change of coordinates
\[
N^{-1} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix},
\]
the plant input matrix \( B \) can be transformed as
\[
\bar{B} = BN^{-1} = \begin{bmatrix} \bar{B}_{11} & \bar{B}_{12} \\ \bar{B}_{21} & \bar{B}_{22} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 2 & -1 \\ 0 & 0 & 2 \end{bmatrix}.
\]

Setting \( \bar{F}_{21} = \begin{bmatrix} -1 & -0.5 & -0.5 \end{bmatrix} \), the matrix \( \bar{F}_{11}^\beta \) has been designed in order to assign stable eigenvalues \( \{-\beta, -2\beta, -3\beta\} \).
with $\beta > 0$ to the annihilator matrix $\tilde{A}^F_{11} = \tilde{A}_{11} + \tilde{B}_{12} \tilde{F}_{21} + \tilde{B}_{11} \tilde{F}_{11}^\beta$. Accordingly, the actuator input $u_{act}$ is defined as

$$u_{act}(s) = y_c(s) + G^-(s)W_{An}(s)v(s) = y_c(s) + \Omega_{An}(s)v(s),$$

where the transfer matrix $W_{An}(s)$ is stable by construction and $v$ is computed by (9) for the cost function $J(z) = \frac{1}{2} |z|^2$.

Three cases have been considered for the allocation gains, namely (note that $\alpha$ is the optimization gain in (9)):

- **Fast allocation**: $\alpha = 40$, $\beta = 0.8$;
- **Slow allocation**: $\alpha = 5$, $\beta = 0.3$;
- **No allocation**: $\alpha = 0$, $\beta = 0.3$.

All initial states are zero. The plant output evolution is depicted in Fig. 6: the three curves, corresponding to the simulated scenarios of fast and slow allocation as well as to the case of unallocated inputs, exactly coincide, this showing that the proposed allocation scheme provides output invisible inputs for weakly-redundant systems. The optimizer performances are illustrated in Fig. 7: with both fast and slow allocation, the steady-state plant input norm is minimized with respect to the case of unallocated control.

**REFERENCES**


