SMC with Linear Dynamic Compensator Design: Performance Margins Approach

Antonio Rosales and Leonid Fridman

Abstract—In this work, the performance of the real sliding mode is characterized with the notion of performance margins under the assumption that the real sliding motion presents a limit cycle with acceptable amplitude and acceptable frequency. The performance of sliding mode controllers, in real sliding motion, is improved by means of the addition of linear compensators in cascade connection with the plant. The applied compensators are phase lead controllers which modify the frequency response of the plant in order to meet the required performance margins adding robustness to the system. Bode diagram interpretation of performance margins is presented for the first time allowing the utilization of SMC combined with the classical methodology of linear compensator design where it is possible shape the Bode diagram in order to obtain desired values of performance margins. Examples supporting the proposed idea are presented.

I. INTRODUCTION

Sliding mode control (SMC) is very efficient in theoretical and practical applications due its insensitivity against matched bounded disturbances and uncertainties [1], [2]. On other hand, the absence of proper models is frequently in most industrial control applications where always exist model uncertainties and external disturbances. Then, SMC technique is the option for tackle this kind of control problems. Real sliding motion is the case of SMC in industrial applications because the incomplete correspondence between the model and the real plant introduces imperfections deteriorating the performance of the controller [1]. Hence, the sliding motion does not occur on the manifold equal zero, but in a neighbourhood of the manifold [1].

The real sliding motion is studied in frequency domain in [3], [2] where the imperfections of SMC performance are interpreted as a limit cycle with finite amplitude and finite frequency on sliding variable. Describing Function (DF) [4] and Locus of a Perturbed Relay System (LPRS) [5] are the techniques used in the frequency domain analysis. Both tools are based in the frequency response of a linear plant controlled by SMC and a solution of Harmonic Balance (HB) equation. Then, by means of DF and LPRS it may be possible predict a limit cycle (amplitude and frequency) on the sliding variable.

Admissible parameters of limit cycle exhibited by the control variable are normally known in control applications. The notions of Performance Phase Margin (PPM) and Performance Gain Margin (PGM) were proposed in [6] under the assumption that these admissible parameters can be defined in terms of an acceptable limit cycle, amplitude and frequency. Hence PPM and PGM are understood as the maximum phase and maximum gain that yields the acceptable amplitude and frequency, respectively. DF technique together HB equation are used in order to identify the performance margins having in this way a measure of the performance of a system under a real sliding mode. Then, once measured both PPM and PGM, these margins could be adequate or not. Whatever the case, the idea of satisfying the desired given performance margins is attractive.

Compensation techniques have been applied to real SMC systems in order to attenuate the performance deterioration. In [7], linear compensators are used in systems with conventional SMC improving non-ideal behavior (real sliding motion) of the system due to unmodeled dynamics. LPRS technique is used for the compensator design. Also via linear compensators, one solution of the same problem of the non-ideal behavior is given in [8] for systems with second order sliding modes (2-SMC), the compensator design is based in DF. Frequency shape techniques using bode diagrams are considered too, for example: in [9] a frequency-shaped SMC consisting of the application of a compensator in order to attenuate the magnitude of the oscillations presented in the control input is proposed; and an application to high-speed-train transportation systems using a linear compensator which shape the bode diagram decreasing the effect of chattering on the output system is reported in [10].

A concept of performance margins that is a measure of system’s robustness to unmodeled dynamics in SMC/2-SMC systems [6] is used in this work not only for the systems robustness analysis, but for the controller design. The idea is to design a cascade linear dynamic compensator, which being combined with the SMC/2-SMC designed for dynamically unperturbed system, guarantees given/desired performance margins.

The contribution of this paper is the following:

- Bode diagram interpretation of performance margins, PPM and PGM, is presented for first time.
- The idea of design linear compensators that satisfy the desired given PPM and PGM of a system comprised of a linear plant controlled by conventional SMC and 2-SMC is proposed.
- A compensator design using Bode diagram allowing the application of classical linear compensator design where one can accomplish the PPM and PGM desired is illustrated.

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Structure of paper is as follows: the problem statement is presented in Section II; Section III contains the identification method of PPM and PGM using Bode diagram for systems controlled by conventional SMC and 2-SMC; performance improvement of SMC systems via the increment of PPM and PGM is illustrated in Section IV; and the conclusions are shown in Section V.

II. PROBLEM STATEMENT

Consider a linear time invariant single-input-single-output plant controlled by SMC. Block diagram of the closed loop system is given in Fig. 1, where the control signal $u$ is generated by SMC and $y$ is the output.

The elements of block-diagram Fig. 1 satisfies the following assumptions:

- **A1.** Assume that the system exhibits a real sliding motion [1], [2] as a limit cycle.
- **A2.** Frequency response $W(j\omega)$ of the SISO linear plant may be obtained.
- **A3.** Linear plant has low pass filter properties, $|W(j\omega)| >> |W(jn\omega)|$ for $n=2, 3, ...$.
- **A4.** Linear plant is strictly proper, i.e., $\lim_{\omega \rightarrow \infty} W(j\omega) = 0$.
- **A5.** The amplitude and phase frequency characteristics of $W(j\omega)$ are monotonously decreasing functions, i.e., $|W(j\omega_1)| > |W(j\omega_2)|$ and $\arg W(j\omega_1) > \arg W(j\omega_2)$ for $\omega_1 < \omega_2$.
- **A6.** Describing Function (DF) $N(A, \omega)$ of SMC may be obtained and it depends only on the amplitude $A$ of self-sustained oscillations $^1$

As shown in [11], the character of unmodeled dynamics in real process is fractal. Accordingly, the relative degree increases and the output $y$ will not converge to zero but to a limit cycle in the real sliding motion [3], [2]. Hence, HB equation gives a solution and limit cycles in systems controlled by SMC may be predicted. Then, tolerance limits presented in the real sliding motion can be defined as follows:

**Definition 1:** The frequency $\omega_c$ and amplitude $A_c$ are the tolerance limits of the acceptable limit cycle of the output $y$, so that self-sustained oscillations of the output $y$ with the amplitudes $A \leq A_c$ and the frequencies $\omega \geq \omega_c$ yield the acceptable performance of the closed loop system in the real sliding mode [12].

The Phase and Gain Margins characterise system’s robustness to unmodeled dynamics [13]. The acceptable performance of the SMC/2-SMC system considered in Fig. 1 can be characterized by Performance Phase Margin and Performance Gain Margin. These margins in such systems are understood in the sense of the following definitions [6]:

**Definition 2 (Performance Phase Margin):** The PPM in system (Fig. 1) is the maximal additional phase shift in $W(j\omega)$ that the closed loop system can tolerate for its output $y$ to exhibit the acceptable predicted limit cycle with $A \leq A_c$, $A_c \geq 0$ and $\omega \geq \omega_c$, $0 < \omega_c < \infty$.

**Definition 3 (Performance Gain Margin):** The PGM in system (Fig. 1) is the maximum additional gain in $W(j\omega)$ that the closed loop system can tolerate for its output $y$ to exhibit the acceptable predicted limit cycle with $A \leq A_c$, $A_c \geq 0$ and $\omega \geq \omega_c$, $0 < \omega_c < \infty$.

The goal of this paper is to guarantee the SMC/2-SMC system’s performance (robustness to unmodeled dynamics) given in terms of Phase and Gain Performance Margins by means of designing a linear cascade compensator that adequately modifies the frequency response $W(j\omega)$ of the SISO plant. This compensator is supposed to operate in a concert with the SMC/2-SMC that are designed for dynamically unperturbed systems. The presence of the linear compensator on the system increases the PPM and PGM. Therefore, the compensated system is more robust against changes of phase and gain on $W(j\omega)$.

III. PERFORMANCE MARGINS: BODE DIAGRAM INTERPRETATION

In this section, it is connected the solution of HB equation with Bode diagrams for first time. Identification methods of PPM and PGM using DF technique with Nyquist plot and HB equation are given in [6]. Now, the graphical identification of PPM and PGM is interpreted by means of Bode plots.

A. **Performance Margins for SMC**

The feedback control system that meets assumptions A1-A6 (see Fig. 1) is considered controlled by the conventional SMC

$$u = -\alpha \cdot \text{sign}(y), \quad (1)$$

where $\alpha > 0$. The HB equation of the system with SMC (1) is

$$W(j\omega) = -\frac{\pi A}{4\alpha} \quad (2)$$

where $W(j\omega)$ is the frequency response of the plant and $\frac{\pi A}{4\alpha}$ is the inverse of DF of SMC depending only of amplitude $A$. Suppose that Bode plot of the linear part is computed. The identification of PPM and PGM with Bode diagram is as follows.
1) Performance Phase Margin: The method to obtain the PPM in Bode diagram consists of 2 steps (see Fig. 2):

Step 1. Locate $A_c$ on the gain plot. It should be done with the right side of equation (2) as

$$A_c = 20 \log \left( \frac{\pi A_c}{4\alpha} \right) \text{ dB}$$

Then, one can find the point $(A_c, \omega_c)$, where $\omega_c$ is the associated frequency of $A_c$. If $\omega_c \geq \omega_c$, then the PPM is identified as

$$PPM_1 = 180 - |\theta_{A_c}|$$

where $\theta_{A_c}$ is the phase associated to $\omega_c$ on the phase plot.

If $\omega_{A_c} < \omega_c$, then one should proceed to Step 2.

Step 2. Locate $\omega_c$ on the gain plot. Then, the point $(A_{\omega_c}, \omega_c)$ is found, where $A_{\omega_c}$ is the associated amplitude of $\omega_c$. Apparently, in this case $A_{\omega_c} \leq A_c$ and the PPM can be found as

$$PPM_2 = 180 - |\theta_{\omega_c}|$$

where $\theta_{\omega_c}$ is the phase associated to $\omega_c$ on the phase plot.

2) Performance Gain Margin: The PGM is identified on the magnitude plot in the frequency associated to the crossing of the phase plot with $-180^\circ$, as it is computed in the classical design. Then, the identification method of PGM consists of the next steps, see Fig. 3:

Step 1: It should be found the frequency $\omega_{GM}$ associated to the intersection of the phase plot with $-180^\circ$.

Step 2: Locate frequency $\omega_{GM}$ on the gain plot. Then, the PGM is identified as

$$PGM = |W(j\omega_{A_c})| - |W(j\omega_{GM})|,$$

where $|W(j\omega_{A_c})|$ is the magnitude evaluated on the frequency associated to the maximal amplitude $A_c$.

3) Example: Suppose, the following linear SISO plant

$$\dot{x}_1 = x_2; \quad \dot{x}_2 = -8x_1 - 3x_2 + u \quad (3)$$

$$y = x_1$$

where the control $u$ is the conventional SMC

$$u = -\text{sign}(y),$$

Fig. 4 presents the Bode diagram of the frequency response $W(j\omega) = \frac{1}{s^2 + 3s + 8}$ of system (3).

Suppose that $A_c = 1 \times 10^{-3}$ and $\omega_c = 50$ [rad/sec]. Then, the PPM and PGM can be identified with the proposed methods as follows: From the right side of equation (2) $A_c = -62.08$ dB and from the Bode plot of Fig. 4 the frequency $\omega_{A_c} \approx 35.8$ [rad/sec]. As $\omega_{A_c} < \omega_c$, it is located the frequency $\omega_c = 50$ [rad/sec]. Hence, it is found the point $(A_{\omega_c}, \omega_c) = (-67.9, 50)$, where $A_{\omega_c} \leq A_c$. Then PPM is computed as follows

$$PPM_1 = 180 - 177 = 3^\circ$$
where $\theta_{Ac} = 177^\circ$ is obtained from Bode plot of Fig. 4.

The frequency response of system (3) does not cross the phase of $-180^\circ$, see Fig. 4. Then, the PGM for the system (3) controlled by conventional SMC is $PGM \to \infty$.

**Remark:** As of Bode diagram interpretation of PPM and PGM, any system with relative degree equal to two, under assumptions A1-A6, controlled by conventional SMC has $PGM \to \infty$ because the phase plot never cross $-180^\circ$. Also, it is satisfied for systems with relative degree equal to one, under assumptions A1-A6.

### B. Performance Margins for 2-SMC

Consider the feedback control system in Fig. 1 that meets assumptions A1-A6 and is controlled by 2-SMC twisting control [2]

$$u = -c_1 \text{sign}(y) - c_2 \text{sign}(\dot{y}), \quad c_1 > c_2 > 0 \quad (4)$$

The DF for 2-SMC (4) is calculated as $N(A) = (4/\pi A)(c_1 + j c_2)$ [14], where $A$ is the amplitude of the predicted limit cycle of the output $y$.

The HB equation for this system takes a form

$$W(j\omega) = \frac{\pi A - c_1 + j c_2}{4 c_1^2 + c_2^2} \quad (5)$$

1) **Performance Phase Margin:** The PPM identification method in Bode diagram for the system of Fig. 1 controlled by Twisting algorithm (4) is similar to the method presented in section IIIA-1 for SMC. The difference is that for Twisting algorithm the inverse negative of its DF (right side of HB equation (5)) does not have a phase equal to $-180^\circ$ as the SMC case. The phase for $-N^{-1}(A)$ in (5) is computed as follows [2]:

$$\arg\left(\frac{-c_1 + j c_2}{c_1^2 + c_2^2}\right) = \arctan\left(\frac{c_2}{c_1}\right)$$

Then, the method for identify PPM is the next.

Step 1. Locate $A_c$ on the gain plot. It should be done with the right side of equation (5) as

$$A_c \equiv 20 \log\left(\frac{\pi A}{4} \sqrt{\frac{c_1^2 + c_2^2}{(c_1^2 + c_2^2)^2}}\right) dB \quad (6)$$

Then, one can find the point $(A_c, \omega_A)$, $\omega_A$ is the associated frequency of $A_c$. If $\omega_A \geq \omega_c$, then the PPM is identified as

$$PPM_1 = 180 + \arctan\left(\frac{c_2}{c_1}\right) - |\theta_{A_c}|$$

where $\theta_{A_c}$ is the phase associated to $\omega_{A_c}$ on the phase plot.

If $\omega_A < \omega_c$ then one should proceed to Step 2.

Step 2. Locate the point $(A_{\omega_c}, \omega_c)$ on the gain plot, $A_{\omega_c}$ is the associated amplitude of $\omega_c$. Apparently, in this case $A_{\omega_c} \leq A_c$ and the PPM can be found as

$$PPM_2 = 180 + \arctan\left(\frac{c_2}{c_1}\right) - |\theta_{\omega_c}|$$

The graphical interpretation of this method can be described by Fig. 2 replacing the reference $-180^\circ$ for $-(180 + \arctan\left(\frac{c_2}{c_1}\right))$ in the phase plot.

2) **Performance Gain Margin:** Now, the PGM is identified on the magnitude plot in the frequency associated to the crossing of the phase plot with $-(180 + \arctan\left(\frac{c_2}{c_1}\right))$.

The method consists of 2 steps:

Step 1: It should be found the frequency $\omega_{GM}$ associated to the intersection of the phase plot with $-(180 + \arctan\left(\frac{c_2}{c_1}\right))$

Step 2: Locate frequency $\omega_{GM}$ on the gain plot. Then, the PGM is identified as

$$PGM_1 = |W(j\omega_{A_c})| - |W(j\omega_{GM})|,$$

where $|W(j\omega_{A_c})|$ is the magnitude evaluated on the frequency associated to the maximal amplitude $A_c$.

Fig. 3 is the graphical interpretation of this method when one replaces the reference $-180^\circ$ for $-(180 + \arctan\left(\frac{c_2}{c_1}\right))$ in the phase plot.

3) **Example:** Let the system (3) controlled by the Twisting algorithm

$$u = -0.8 \text{sign}(y) - 0.6 \text{sign}(\dot{y})$$

Bode plot of the frequency response $W(j\omega) = 1/(s^2 + 3s + 8)$ of system (3) is presented in Fig. 5.

Also, $A_c = 1 \times 10^{-3}$ and $\omega_c = 50$[rad/sec]. The PPM is identified with the next steps: First, the point $(A_c, \omega_A)$ is obtained from equation (6) and Bode diagram Fig. 5. Because $\omega_A < \omega_c$, now it is located the point $(A_{\omega_c}, \omega_c)$ and the PPM is

$$PPM = 180 + \arctan\left(\frac{0.6}{0.8}\right) - 177 = 39.87^\circ$$

where $\theta_{\omega_c} = 177^\circ$ is obtained from Bode plot of Fig. 5.

The $PGM \to \infty$ for system (3) because the phase plot never cross $-(180 + \arctan\left(\frac{c_2}{c_1}\right))$.

**Remark:** As of Bode diagram interpretation of PPM and PGM, any system with relative degree equal to two, under...
assumptions A1-A6, controlled by 2SMC has PGM → ∞ and PPM ≥ arctan \( \frac{c_2}{c_1} \) ° because the phase plot never cross −180° and it goes asymptotically to −180°.

Once the identification of PPM and PGM was presented, the way of satisfying the desired given performance margins of SMC controllers via linear compensators is shown in the next section.

IV. PERFORMANCE IMPROVEMENT VIA LINEAR COMPENSATOR

The used linear compensator is a phase lead filter represented by the next transfer function:

\[ W_c = \frac{s + \frac{1}{\beta \tau}}{s + \frac{1}{\tau}}; 0 < \beta < 1 \quad (7) \]

The compensator design is based on the classical methodology using Bode diagram [13]. The objective is satisfying the desired performance margins of the SMC/2-SMC system via the linear compensator (7) in order to add robustness to the compensated system against changes of phase and gain on \( W(j\omega) \).

A. Compensator Design

The design of the compensator (7) for systems controlled by SMC is illustrated in the next example.

1) Example Conventional SMC system: Consider the example presented in subsection III.A-3. Recall that the plant (3) is controlled by SMC \((u = -sign(y))\). Suppose that a delay \( T = 0.01 \) seconds emerges. Now, the transfer function of the system is

\[ W_a = \frac{e^{-0.01s}}{s^2 + 3s + 8} \quad (8) \]

The desired PPM and PGM are \( PPM_c > 35^\circ \) and \( PGM_c > 10 \) dB, respectively, in spite of time delay \( T \). Then, the objective is design a compensator satisfying the requirements \( PPM_c \) and \( PGM_c \).

Step 1. Performance margins of uncompensated system should be obtained. \( PGM = -12.48 \) dB and \( PPM = -16^\circ \) of the plant (3) with delay \( T = 0.01 \) are shown in Fig. 6. One can see, that the phase plot crosses −180° before the magnitude plot crosses the value associated to \( A_c \). Therefore, the PPM and PGM are negative and the system does not have acceptable performance \( PPM_c > 35^\circ \) and \( PGM_c > 10 \) dB. Due to uncompensated system does not satisfy the \( PPM_c > 35^\circ \) and \( PGM_c > 10 \), one should be proceeded to the next step.

Step 2. Maximum phase of (7) is computed as

\[ \phi_m = PPM_c^\circ - PPM_{un}^\circ + (5, 12)^\circ, \]

where \( PPM_{un} \) is the \( PPM \) of the uncompensated system and \( (5, 12) \) means a interval. For this example

\[ \phi_m = 35^\circ + 10^\circ + 10^\circ = 61^\circ \]

Step 3. Parameter \( \alpha \) is computed by means of

\[ \sin \phi_m = \frac{1 - \beta}{1 + \beta} \]

\[ \beta = 0.0669 \]

Step 4. It is identify the magnitude where the uncompensated system is

\[ 20 \log \left( \frac{1}{\sqrt{\beta}} \right) + 20 \log \left( \frac{\pi A_c}{4\alpha} \right), \]

and its frequency associated \( \omega_m \). In our example

\[ 20 \log \left( \frac{1}{\sqrt{0.0669}} \right) + 62.08 \text{ dB} = 73.83 \text{ dB}, \]

from Bode plot of uncompensated system (Fig. 6) the point \((|W_a| = -73.83 \text{ dB}, \omega_m = 70.2 \text{ rad/sec})\) is located

Step 5. The pole and zero of \( W_c \) are computed

\[ \text{Pole : } \frac{1}{\tau} = \omega_m \sqrt{\beta} \]

\[ \text{Zero : } \frac{1}{\beta \tau} \]

For the example

\[ \frac{1}{\tau} = 78.08 \sqrt{0.0251} = 18.15 \]

\[ \frac{1}{\beta \tau} = \frac{1}{0.0251} (12.37) = 271.3 \]

Finally the compensator is

\[ W_{c1} = \frac{s + 18.15}{s + 271.3}. \quad (9) \]

The compensated system has \( PGM = 27.62 \) dB and \( PPM = 39^\circ \), see Fig. 7. Hence, the compensated system satisfies the requirements of PPM and PGM. Actually, the system with no acceptable performance was become a system with both acceptable performance and wished PPM and PGM via the compensator. Moreover, due to the increment of PPM and PGM, the compensated system is more robust against phase and gain changes.
Then, it is necessary to design a compensator. Because uncompensated system does not have the desired performance. SMC controlled by

\[ u = 54 \text{ V} \]

is obtained. The results of the compensated system are shown in Fig. 9.

The compensated system has \( PGM = 20.42 \text{ dB} \) and \( PPM = 54.87^\circ \), see Fig 9. Therefore, the desired performance is reached by means of the addition of compensator. Moreover, the compensated system is more robust against phase and gain changes than the uncompensated system.

### V. Conclusions

The relation between HB equation and Bode diagrams, studied for the first time, is utilized for the identification of Practical Phase Margin and Practical Gain Margin. The desired given performance margins are guaranteed in systems with SMC and 2-SMC by means of the adding the linear compensators in cascade with 2-SMC/SMC controllers. The compensated system features the desired given PPM and PGM. Therefore, the compensated system is guaranteed the desired robustness against phase and gain changes on plant. The proposed results provide the useful robust control design tools for a variety of control applications.

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## REFERENCES


