Ensuring Stability in Continuous Time System Identification
Instrumental Variable Method For Over-parameterized Models

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Abstract—The aim of this paper is to develop constraints to ensure stability of the model in the continuous time, simplified refined instrumental variable system identification algorithm (SRIVC) for over-parameterized models. Specifically, a convex stability domain in the space of polynomial coefficients will be generated and the system parameters will be estimated within this domain. It is found that the model fit obtained using the proposed method offers an improvement to the typical SRIVC method. A Monte Carlo simulation is presented to illustrate the performance of the proposed approach.

Keywords: Continuous time identification; instrumental variable methods; least squares.

I. INTRODUCTION

Over the last few decades, the field of continuous time (CT) system identification has grown both rapidly and strongly. The first significant survey of this field was presented by Young in 1981 [14]. Following that, there are many papers and books have been dedicated to the subject of CT system identification [5] [7] [10] [11] [13]. Among all CT identification methods, the refined instrumental variable method (RIVC) and its simplified version (SRIVC) are well accepted. They were first proposed by Young and Jakeman [16], and have been successfully used in many practical applications [15] [17]. These algorithms are also included in the CONTSID [8] and CAPTAIN [18] toolboxes.

In the refined instrumental variable methods, there is an underlying issue in that the stability of the estimated system is not guaranteed when the model order is not chosen correctly, e.g. over-parameterization or when the data is very poor. Until now, the only method developed to address this issue is to reflect the unstable eigenvalues of the estimated transfer function denominator into the stable region of the complex plane i.e. left half plane [9]. The advantage of this method is that it ensures the magnitude of the estimated transfer function (TF) to be the same before and after reflection. However, it inherently alters the phase of the estimated TF.

To overcome this issue, we propose to generate a stability domain in the space of polynomial coefficients, then estimate system parameters by using the SRIVC method constrained by this domain. There are several popular stability criterion such as Routh-Hurwitz theorem [1] [3] or Kharitonov criterion [19] however the stability domain generated by these methods is non-convex in general. Essentially, this is why several convex approximations of the stability region such as ellipsoids [2] [4], hyperrectangles [6] [19] and polytopes [12] are well-known and widely used in robust control.

For some complex systems, the uncertainty in model structure selection can lead to an underestimation of the system. In that case, an over-parameterized model is useful to avoid any kind of underfitting. In this paper, we specifically consider the case when over-parameterization causes instability of estimated system. We then propose a modification to the SRIVC method that forces the estimated system to be stable by the use of an ellipsoid approach. First, the original SRIVC is used to find an initial estimate, then a convex stability domain is generated using these initial parameters. Finally, we solve the problem using the SRIVC method constrained by this domain.

The paper is organised as follows. Section II describes the model setting and Section III recalls the methodology of the SRIVC algorithm. Section IV establishes the algorithm of the new method and discusses the effectiveness of this modification. The numerical experiment and results will be provided in Section V. Finally, section VI will present the conclusion.

II. MODEL SETTING

Consider the continuous linear, time invariant, input-output system,

\[ x(t) = G(p)u(t) = \frac{B(p)}{A(p)}u(t), \]

with

\[ B(p) = b_0p^n + b_1p^{n-1} + \ldots + b_m, \]
\[ A(p) = p^n + a_1p^{n-1} + \ldots + a_n, \quad n \geq m \]

where \( x(t) \) is the deterministic output of the system; \( p \) is the differential operator, i.e. \( p^i x(t) = \frac{d^ix(t)}{dt^i} \). Furthermore, \( B_0(p) \) and \( A_0(p) \) are assumed to be coprime and the system is asymptotically stable.

The output \( x(t) \) is observed at \( t = t_k \) by \( y(t_k) \) as

\[ y(t_k) = x(t_k) + v(t_k), \]

where \( v(t_k) \) is a noise sequence.

The objective then is to estimate the parameters \( a_1, a_2, \ldots, a_n \) and \( b_0, b_1, \ldots, b_m \) of the CT model (1), based on the discrete input and output data \( u(t_k), y(t_k)_{k=1}^N \).
III. SRIVC ALGORITHM

In this section we recall the SRIVC method as developed in the literature by Young and Garnier [7][9][14]. From (1) and (2), we have

\[ y(t_k) = \frac{B(p)}{A(p)} u(t_k) + v(t_k), \]

which can be expressed as,

\[ \frac{A(p)}{A(p)} y(t_k) = \frac{B(p)}{A(p)} u(t_k) + v(t_k). \]  

(4)

Then by denoting

\[ y_A(t_k) = \frac{1}{A(p)} y(t_k), \]

\[ u_A(t_k) = \frac{1}{A(p)} u(t_k), \]

we have,

\[ A(p) y_A(t_k) = B(p) u_A(t_k) + v(t_k). \]

(6)

This gives a linear regression model,

\[ y_A^{(n)}(t_k) = \varphi_N^T(t_k) \theta + v(t_k) \]

(7)

where

\[ \varphi_N(t_k) = [-y_A^{(n-1)}(t_k), \ldots, -y_A(t_k), \]

\[ u_A^{(m)}(t_k), u_A^{(m-1)}(t_k), \ldots, u_A(t_k)], \]

(8)

\[ \theta = [a_1, a_2, \ldots, a_n, b_0, b_1, \ldots, b_m]^T, \]

(9)

\[ y_A^{(i)}(t_k) = \frac{d^i y_A(t_k)}{dt^i}. \]

(10)

Notice that \( y_A(t_k) \) and \( u_A(t_k) \) represent the sampled outputs of the continuous-time pre-filtering operations,

\[ f_p = \frac{1}{A(p)}. \]

(11)

The signals \( y_A(t_k) \) and \( u_A(t_k) \) and their derivatives can be computed by calculating the states of the state space model in (12) driven by \( y(t_k) \)

\[
\begin{bmatrix}
  y_A^{(n)}(t_k) \\
  y_A^{(n-1)}(t_k) \\
  \vdots \\
  y_A^{(1)}(t_k) \\
\end{bmatrix} =
\begin{bmatrix}
  -a_1 & -a_2 & \ldots & -a_n \\
  1 & 0 & \ldots & 0 \\
  \vdots & \vdots & \ddots & \vdots \\
  0 & \ldots & 0 & 1 \\
\end{bmatrix}
\begin{bmatrix}
  y_A^{(n-1)}(t_k) \\
  y_A^{(n-2)}(t_k) \\
  \vdots \\
  y_A(t_k) \\
\end{bmatrix}
+ \begin{bmatrix}
  1 \\
  0 \\
  \vdots \\
  0 \\
\end{bmatrix} y(t_k)
\]

(12)

The associated optimal IV matrix \( \hat{\varphi}_A(t_k) \) is an estimate of the noise-free version of the matrix \( \varphi_A(t_k) \) in (7) and is defined as,

\[ \hat{\varphi}_N(t_k) = [-x_A^{(n-1)}(t_k), \ldots, -x_A^{(n-2)}(t_k), \ldots, -x_A(t_k), u_A^{(m)}(t_k), u_A^{(m-1)}(t_k), \ldots, u_A(t_k)]. \]

(13)

where

\[ x_A(t_k) = \frac{1}{\hat{A}(p)} x(t_k); \]

\[ \hat{B}(p) \) and \( \hat{A}(p) \) are estimates of \( B(p) \) and \( A(p). \)

The SRIVC method can be summarized as [9].

Step 1. Initialisation

1) A stable filter is chosen in order to generate derivatives of \( y(t_k) \) and \( u(t_k) \). The break point parameter \( \lambda \) (break point frequency in radians/sec) is chosen to be equal to or larger than, the bandwidth of the system to be identified,

\[ f_p = \frac{1}{(p + \lambda)^n}. \]

(15)

2) Filter \( y(t_k) \) and \( u(t_k) \) via the chosen stable filter in order to compute the derivatives of the signals \( y_A^{(n)}(t_k), y_A^{(n-1)}(t_k), \ldots, y_A(t_k) \) and \( u_A^{(m)}(t_k), u_A^{(m-1)}(t_k), \ldots, u_A(t_k) \) (see Equation (12))

3) Use the least squares method to estimate the initial parameters based on equation (7). The initial estimated TF after this step is \( \hat{G}(p, \theta^0) \) where,

\[ \hat{\theta}^0 = [\Phi_N \Phi_N^T]^{-1} \Phi_N Y_N. \]

(16)

with

\[ \Phi_N^T = [\varphi_N(t_1) \ldots \varphi_N(t_N)]^T, \]

\[ Y_N = [y_A^{(n)}(t_1) y_A^{(n)}(t_2) \ldots y_A^{(n)}(t_N)]^T. \]

(17)

(18)

Step 2. Iterative estimation

for \( j = 1: convergence \)

1) Generate the instrumental variable series \( \hat{x}(t_k) \) using the current estimated system \( \hat{G}(p, \hat{\theta}^{j-1}) \), i.e.

\[ \hat{x}(t_k) = \frac{\hat{B}(p, \hat{\theta}^{j-1})}{\hat{A}(p, \hat{\theta}^{j-1})} u(t_k). \]

(19)

2) Prefilter the input \( u(t_k) \), output \( y(t_k) \) and the instrumental variable \( \hat{x}(t_k) \) by the continuos time filter

\[ f_c(p) = \frac{1}{\hat{A}(p, \hat{\theta}^{j-1})}. \]

(20)

3) Based on the prefiltered data, generate an estimate \( \hat{\theta}^j \) of the system parameters using the IV method,

\[ \hat{\theta}^j = [\Psi_N \Phi_N^T]^{-1} \Psi_N Y_N. \]

(21)

where \( \Psi_N \) is the IV matrix generated by the instrumental variables \( \hat{x}(t_k) \) and \( \hat{\Phi}_N \) is the regression matrix of the linear problem in (7),

\[ \Phi_N^T = [\varphi_N(t_1) \ldots \varphi_N(t_N)]^T, \]

(22)
\( \Psi_N^T = [\hat{\phi}_N(t_1) \ldots \hat{\phi}_N(t_N)]^T, \)
\( Y_N = [y_A^{(n)}(t_1) y_A^{(n)}(t_2) \ldots y_A^{(n)}(t_N)]^T. \)

4) If the estimated TF model is unstable, reflect the unstabeleigenvalues of the estimated \( \hat{A}(p, \hat{\theta}) \) polynomial into the stable region of the complex plane and construct the new estimator. \(^1\)

end

IV. SRIVC STABILITY CONSTRAINING ALGORITHM

Our objective here is to generate a convex stability domain in the space of polynomial coefficients and then use the SRIVC method to estimate the system within this domain.

First, we quote a theorem from [2] as Lemma 1. This result facilitates the development of the method proposed in this paper.

**Lemma 1**

Let

\[ D = s \in \mathbb{C} : a + b(s + \bar{s}) + cs\bar{s} < 0 \]

be a given stability region of the complex plane, where \( a, b, c \in \mathbb{R} \). Let

\[
\begin{bmatrix}
  x_0 \\
  x_1 \\
  \vdots \\
  x_{n-1} \\
  x_n
\end{bmatrix}
\]

be vectors of real coefficients of the polynomial

\[ x(s) = x_0 + x_1s + \ldots + x_{n-1}s^{n-1} + x_n s^n. \]

It is assumed without loss of generality that \( x(s) \) is monic, i.e., \( x_n = 1 \).

Now let \( H \) be the associated matrix of the stability region \( D \), which is computed by:

\[
(I_n \otimes x)^T H (I_n \otimes x) = H(x)
\]

(25)

where \( H(x) \) is the Hermite matrix of polynomial \( x(s) \). It can be obtained from equations (26) and (27),

\[
xx^T - \bar{x}x^T = aR_l^T H(x)R_l + b(R_l^T H(x)R_r + R_r^T H(x)R_l) + cR_r^T H(x)R_r,
\]

(26)

\[
\hat{x}(s) = \left( \frac{b + cs}{\sqrt{b^2 - ac}} \right)^n x \left( \frac{-a + bs}{b + cs} \right),
\]

(27)

\[
R_l = \begin{bmatrix}
  1 & 0 & \ldots & 0 \\
  0 & 1 & \ldots & 0 \\
  \vdots & \vdots & \ddots & \vdots \\
  0 & 0 & \ldots & 1
\end{bmatrix}, \quad
R_r = \begin{bmatrix}
  0 & 1 & \ldots & 0 \\
  0 & 0 & \ldots & 0 \\
  0 & 0 & \ldots & 0 \\
  0 & 0 & \ldots & 1
\end{bmatrix}
\]

(\( R_l \) and \( R_r \) are projection matrices of size \( n \times (n + 1) \))

The lemma states that for any arbitrary vector \( x_C \) (that forms a stable polynomial \( x(s) \)), then any positive definite matrix \( P \) that satisfies the following convex optimisation problem will have the following property: Any vector \( x \) that’s in the ellipsoid \( (x - x_C)^T P(x - x_C) \leq 1 \) will parametrize a polynomial \( x(s) \) with all its roots in \( D \).

**The convex optimisation problem we consider is then**

\[
\begin{align*}
\text{maximize} & \quad \text{trace}(P_{11}) \\
\text{s.t.} & \quad (D \otimes I_{n+1}) H > I_n \otimes P + G
\end{align*}
\]

(31)

where

(a) \( P \) is a symmetric matrix which is partitioned as

\[
P = \begin{bmatrix}
  P_{11} & P_{12} \\
  P_{T12}^T & P_{22}
\end{bmatrix}
\]

(28)

with

\[
P_{11} = -P, \quad P_{12} = Px_C, \quad P_{22} = 1 - x_C^T Px_C.
\]

(29)

(b) \( G \) is a symmetric block matrix

\[
G = \begin{bmatrix}
  0 & G_{21}^T & \ldots & G_{n1}^T \\
  G_{21} & 0 & \ldots & G_{n2}^T \\
  \vdots & \vdots & \ddots & \vdots \\
  G_{n1} & G_{n2} & \ldots & 0
\end{bmatrix}
\]

(30)

made up of skew-symmetric matrices \( G_{ij} = -G_{ji} \in \mathbb{C}^{(n+1)\times (n+1)} \).

(c) and \( D \in \mathbb{R}^{n \times n} \) is a symmetric matrix satisfying

\[
D > 0, (D \otimes I_{n+1}) H = H(D \otimes I_{n+1})
\]

If \( D \) is the left half plane, then \( a = 0, b = 1, c = 0 \) in (26) and (27).

From the lemma 1 and simple method of reflecting the unstable poles, we can make some general points:

- For any arbitrary vector \( x_C \) (that forms a stable polynomial \( x(s) \)), we can define a convex domain in the space of polynomial which has \( x_C \) as the center of the domain.

- The reflecting-pole method already generates a stable estimated transfer function. The inaccuracy of this method is caused by the phase changes when reflecting the poles. This can be reduced if the poles are moved to a location with better phase and maintaining the same magnitude.

Based on these observations, next we develop the algorithm for the continuous time, simplified instrumental variable based system identification which forces stable estimates in all iterations.

**Stage 1. Initialisation**

Follow all the steps in the normal SRIVC algorithm as listed above.

\(^1\)In this paper, we put step 4) as the last step so that later we can describe the proposed method easier. This step can be put at the first step in the iteration [9].
Stage 2. Iterative estimation
for $j = 1$: convergence
From steps 1 to 3, follow as in the normal SRIVC algorithm listed above to find
\[
\hat{G}(p, \hat{\theta}^j) = \frac{\hat{B}(p, \hat{\theta}^j)}{\hat{A}(p, \hat{\theta}^j)} = \frac{\hat{b}_0 p + \hat{b}_1 p^{-1} + \cdots + \hat{b}_n}{\hat{a}_0 p^n + \hat{a}_1 p^{n-1} + \cdots + \hat{a}_n} \quad (32)
\]
4) Calculate poles of the estimated transfer function $G(p, \hat{\theta}^j)$, reflect all the poles that are in the right half plane to the left half plane and then recalculate the denominator polynomials $\hat{A}(p, \hat{\theta}^j)$. After this step, $\hat{A}(p, \hat{\theta}^j)$ will have this form,
\[
\hat{A}(p, \hat{\theta}^j) = p^n + \hat{a}_{new,1} p^{n-1} + \cdots + \hat{a}_{new,n} \quad (33)
\]
5) Choose $x_C = [\hat{a}_{new,n} \hat{a}_{new,n-1} \cdots \hat{a}_{new,1}]^T$ as the center of the ellipsoid, calculate the matrix $P$ following Lemma 1.
6) Solve the following convex optimization problem to find new $\hat{\theta}^j$,
\[
\text{minimize}_{\theta} \quad \| \Psi_N Y_N - \Psi_N \Phi_N^T \theta \|^2 \\
\text{s.t.} \quad \theta(n : 1) - x_C)^T P(x(n : 1) - x_C) \leq 1 \\
\theta(n) = \theta(n + m + 1) \frac{\hat{a}_n}{\hat{b}_n}
\]
where $\Psi_N, Y_N, \Phi_N$ are the matrices in step 3 (21), $P$ is the matrix calculated from step 5 and $\hat{a}_n, \hat{b}_n$ are the polynomial coefficients in (32)
end

Discussion
Let us denote $G(s)$ as a standard rational transfer function with $m$ zeros and $n$ poles.
\[
G(s) = \frac{c(s - z_1)(s - z_2) \cdots (s - z_{m-1})(s - z_m)}{(s - p_1)(s - p_2) \cdots (s - p_{n-1})(s - p_n)} \quad (34)
\]
Then for any point on the imaginary axis,
\[
\|G(jw)\| = |c| \sqrt{\frac{(w^2 + z_1^2)(w^2 + z_2^2) \cdots (w^2 + z_m^2)}{(w^2 + p_1^2)(w^2 + p_2^2) \cdots (w^2 + p_n^2)}} \quad (35)
\]
and
\[
\angle G(jw) = \angle c + \sum_{i=1}^{m} \angle(jw - z_i) - \sum_{i=1}^{n} \angle(jw - p_i). \quad (36)
\]
There are two distinct situations that can occur for the pole locations.

Case 1: The unstable pole does not have any zero close to it
In Figure 1, let B be the original pole location before reflecting to the left half plane, C be the pole location after reflection and D be a pole location which is nearer the imaginary axis than C.
After reflection, the magnitude of $G(jw)$ is the same however the phase of $G(jw)$ is changed,
\[
\Delta \angle G(jw) = (-\alpha_1) - (-\alpha_2) = \pi - 2\alpha_1
\]
When the new pole location is D, the difference between the original phase and the new phase is
\[
\Delta \angle G(jw) = (-\alpha_1) - (-\alpha_3) = \alpha_3 - \alpha_1
\]
D is nearer the imaginary axis than C, so
\[
\alpha_3 > \alpha_2 \\
\Rightarrow \alpha_1 + \alpha_3 > \alpha_1 + \alpha_2 = \pi \\
\Rightarrow 2\alpha_1 - \pi > \alpha_1 - \alpha_3 \\
\Rightarrow |\pi - 2\alpha_1| > |\alpha_3 - \alpha_1|
\]
\[
\Rightarrow \text{the phase difference between the original pole location and the new location is smaller when the new pole location is closer to the imaginary axis}
\]
However, when the new pole location moves nearer the imaginary axis, the magnitude will change because this new location and the original location are not symmetric through the imaginary axis. This is a trade-off between the phase and magnitude of the transfer function. In the proposed method, by solving the optimization problem in step 6 of the algorithm, a pole location will be found to increase the accuracy of the estimated transfer function whilst maintaining stability. In the simulation experiment we conduct (described in the next section), the accuracy of the estimates is improved over the original method.

Case 2: The unstable pole has a zero close to it
In Figure 2 and Figure 3, let B be the original pole location before reflecting to the left half plane, C be the pole location after reflection, D be a pole location which is nearer the imaginary axis than B. Let Y be the original zero location before reflecting the pole and Z be a zero location that is near D.
The difference between the original phase and new phase using the reflection method is,
\[
\Delta \angle G(jw) = (\beta_1 - \alpha_1) - (\beta_1 - \alpha_3) = \alpha_3 - \alpha_1 \quad (37)
\]
The difference between the original phase and new phase using the proposed method is,
\[
\Delta \angle G(jw) = (\beta_1 - \alpha_1) - (\beta_2 - \alpha_2) = \beta_1 - \alpha_1 + \alpha_2 - \beta_2 \quad (38)
\]
In this section, a numerical example is presented to demonstrate the veracity of the proposed method. The test system is the Rao-Garnier Test System [7]:

$$G(s) = \frac{-6400p + 1600}{p^3 + 5p^3 + 408p^2 + 416p + 1600}.$$  

This test system is a linear, non-minimum phase system with complex poles. A PRBS (pseudo-random binary sequence) of maximum length is used as an input to excite the system over its entire bandwidth, the sampling time $T_s$ is chosen as 10ms, the number of stages of the shift register is set to 10, the clock period is set to 7, which results in the number of samples, $N = 7161$.

In the experiment, the noise process is assumed to be white noise with zero mean and Signal to Noise Ratios (SNR) to be approximately 10dB and 20dB, respectively.

The estimated model is chosen to have order 6, i.e. $n = 6, m = 5$ (over-parameterized model). Identification of this model was performed in 1000 Monte-Carlo runs using the original SRIVC method (from the CONTSID toolbox) and the modified method as proposed in this paper.

To evaluate the two methods, we utilise a model fit score in the following formulation,

$$W = 100 \left( 1 - \frac{\sum_{k=1}^{\infty} (h_k - \hat{h}_k)^2}{\sum_{k=1}^{\infty} (h_k - \text{mean}(h_k))^2} \right)^{1/2} \quad (39)$$

where $h_k$ is the impulse response of the real system and $\hat{h}_k$ is impulse response of the estimated system.

Note that when $W = 100$ the estimated system fits perfectly with the real system.

The results of the two numerical experiments are provided in table (I). In summary, for SNR = 20dB, 975/1000 runs required stabilisation; and for SNR = 10dB, 981/1000 runs required stabilisation.

<table>
<thead>
<tr>
<th>SNR</th>
<th>Indicators</th>
<th>Original SRIVC method</th>
<th>Proposed method</th>
</tr>
</thead>
<tbody>
<tr>
<td>10dB</td>
<td>Mean</td>
<td>95.9116</td>
<td>96.6422</td>
</tr>
<tr>
<td></td>
<td>Median</td>
<td>98.0813</td>
<td>98.3094</td>
</tr>
<tr>
<td></td>
<td>Variance</td>
<td>42.3927</td>
<td>22.8146</td>
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<td></td>
<td>Min</td>
<td>39.2630</td>
<td>38.4018</td>
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<tr>
<td></td>
<td>Max</td>
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<td>99.5004</td>
</tr>
<tr>
<td>20dB</td>
<td>Mean</td>
<td>96.5019</td>
<td>97.1360</td>
</tr>
<tr>
<td></td>
<td>Median</td>
<td>98.6039</td>
<td>99.4752</td>
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<tr>
<td></td>
<td>Variance</td>
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<td></td>
<td>Min</td>
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<tr>
<td></td>
<td>Max</td>
<td>99.2815</td>
<td>99.8266</td>
</tr>
</tbody>
</table>
Boxplots of fit scores from the experiment with white noise and SNR = 10dB, SNR = 20dB are also displayed in Figure 4 and Figure 5 to compare the accuracy between two methods.

In addition, pole zero maps of an estimated transfer function before and after reflection and by using new method are plotted in Figure 6 to show the effect of the proposed method on relocating the poles.

Based on this experiment, we see that the significant effect of the proposed method is to improve all the summary statistics of the fit score such as mean, median, variance ... in all the situations we considered, i.e. any noise level, small or large SNR.

VI. Conclusion

In this paper we have developed an algorithm to ensure that a model estimated by the simplified refined instrumental variable method is stable. The proposed method is shown to provide better estimate than the original SRIVC algorithm by a Monte Carlo study of the estimates under different noise conditions. In particular, we have considered the case of when the model order is unknown and the system is over modelled.

References