Barrier function based linear model predictive control with polytopic terminal sets

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Abstract—We present a novel approach for the stabilizing design of barrier function based linear model predictive control schemes. In contrast to existing works, the approach is based on polytopic terminal sets and a smooth approximation of the corresponding Minkowski functional. We prove asymptotic stability of the closed-loop system and discuss potential advantages of the proposed design procedure. The results are illustrated by means of a numerical example.

I. INTRODUCTION

Model predictive control (MPC) is a modern control concept for constrained linear and nonlinear systems, in which the control input is obtained by solving on-line at each sampling instant a suitable finite-horizon open-loop optimal control problem that is parametrized by the current system state. Only the first element of the optimal control input sequence is applied to the plant and the optimization is repeated at the next time step. Besides a user-defined cost objective, both the dynamics of the system to be controlled and potential constraints on the system states and inputs can be taken into account in the underlying optimization problem. There exists various results and concepts concerning stability properties of the closed loop as well as efficient algorithmic implementations for MPC of linear and nonlinear systems, see [1], [2], [3], [4], [5], [6], [7]. Due to these reasons, MPC is a widely accepted and powerful control concept that is also more and more applied to industrial processes, see [8] and references therein.

In barrier function based MPC approaches, the inequality constraints occurring in the open-loop optimal control problem are eliminated by incorporating them into the cost function by means of suitable barrier function terms with a corresponding weighting factor [9]. In this way, the resulting optimization problem is merely equality constrained (or unconstrained after elimination of the linear system dynamics) and can be tackled by tailored optimization techniques. In addition, as pointed out in [9], the barrier function weighting can be seen as a safety parameter that provides some degree of controller caution near constraint boundaries. Interestingly, it is moreover possible to design continuous-time algorithms that asymptotically track the optimal solution of a barrier function based open-loop optimal control problem formulation and thus allow to implement MPC without any iterative on-line optimization, see [3], [5], [10], [11]. Two different approaches towards the stabilizing design of barrier function based MPC schemes have been presented in [9] and [10] based on ellipsoidal terminal sets. Furthermore, an extension of these results to the case of so-called relaxed logarithmic barrier functions has been proposed in [11].

In the context of linear discrete-time systems, however, the question naturally arises whether one could also make use of terminal sets that are given in form of polytopes. For one thing, polytopic terminal sets are directly compatible with the natural representation of (maximal) positively invariant sets of linear systems. Furthermore, compared for example to ellipsoids, polytopic representations are more flexible and might allow to formulate larger terminal sets especially in the presence of asymmetric state and input constraints. However, ensuring stability properties of barrier function based MPC schemes when using polytopic terminal sets is a nontrivial, and to the knowledge of the authors open, problem. In order to allow for an efficient solution of the resulting open-loop optimal control problem by means of standard convex optimization procedures or the outlined continuous-time MPC algorithms, we are moreover especially interested in a smooth and convex formulation of the underlying cost function.

Motivated by this, we want to discuss in this paper the questions whether and how we can design stabilizing barrier function based MPC schemes that make use of polytopic terminal sets and how we can formulate the underlying cost function in a smooth way. We will answer the first question by presenting an asymptotically stabilizing barrier function based MPC approach which makes use of the so-called Minkowski functional and its properties for positively invariant sets. Furthermore, we present a second approach in which the nonsmooth Minkowski functional is approximated by a smooth, convex, and positive definite function and we prove asymptotic stability also for this setup. Hence, the contribution of this paper is to present a first approach towards the design of asymptotically stabilizing barrier function based MPC formulations that make use of polytopic terminal sets and can, due to their smooth and convex formulation, be solved efficiently by both conventional and continuous-time optimization algorithms.

Throughout the paper, we will make use of the following notation. Positive definiteness or positive semi-definiteness of a symmetric matrix $M$ is denoted as $M \succ 0$ and $M \succeq 0$, respectively; we define $\|x\|_M := \sqrt{x^\top M x}$ for any $M \succeq 0$; the expression $M_i$ refers to the $i$-th row of a given matrix $M$; for any arbitrary set $S$, $S^\circ$ will denote the open interior and $\partial S$ the limiting boundary.

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II. BACKGROUND AND PROBLEM SETUP

A. Preliminaries

Let us consider a discrete-time linear system of the form
\[ x(k+1) = Ax(k) + Bu(k), \]  
where \( x(k) \in \mathbb{R}^n \) and \( u(k) \in \mathbb{R}^m \) refer to the vectors of system states and inputs at time instant \( k \) and the real matrices \( A \in \mathbb{R}^{n \times n} \) and \( B \in \mathbb{R}^{n \times m} \) with \((A,B)\) stabilizable describe the underlying linear system dynamics. Assume now that our goal is to regulate the system state to the origin while satisfying state and input constraints of the form \( x(k) \in \mathcal{X} \) and \( u(k) \in \mathcal{U} \) for all discrete sampling points \( k \geq 0 \). Here, \( \mathcal{X} \subset \mathbb{R}^n \) and \( \mathcal{U} \subset \mathbb{R}^m \) are given convex sets that contain the origin in their interior. In MPC, this control problem is usually tackled by solving at each sampling instant a finite-horizon open-loop optimal control problem of the form

\[
J^*(x) = \min_u \sum_{k=0}^{N-1} \ell(x_k, u_k) + F(x_N) \tag{2a}
\]

subject to

\[
\begin{align*}
& x_{k+1} = Ax_k + Bu_k, \quad x_0 = x := x(k) \tag{2b} \\
& x_k \in \mathcal{X}, \quad k = 0, \ldots, N-1, \quad x_N \in \mathcal{X}_f \subset \mathcal{X}, \tag{2c} \\
& u_k \in \mathcal{U}, \quad k = 0, \ldots, N-1, \tag{2d}
\end{align*}
\]

over a given finite prediction horizon \( N \) and applying the control law \( u(k) = u^*_Q(x(k)) \) in a receding horizon fashion. Here, the stage cost \( \ell(x,u) \) and the terminal cost \( F(x) \) are defined as
\[
\ell(x,u) = \|x\|^2_Q + \|u\|^2_R \quad \text{and} \quad F(x) = \|x\|^2_P, \tag{3a}
\]

for appropriately chosen weight matrices \( Q = Q^\top \succeq 0, R = R^\top > 0, P = P^\top > 0 \). The optimization is performed over the open-loop input sequence \( u := \{u_0, \ldots, u_{N-1}\} \) and \( \mathcal{X}_f \) refers to a closed and convex terminal constraint set that is used to guarantee stability properties of the closed-loop system. More details on conventional MPC schemes for linear and nonlinear systems as well as theoretical results on stability and recursive feasibility can be found in [1].

B. Barrier function based model predictive control

The main idea in barrier function based MPC is to eliminate the inequality constraints from the above MPC open-loop optimal control problem by making use of suitable barrier functions. Based on this idea, which is the core of the well-known interior-point methods, it is possible to reformulate problem (2) as an unconstrained, respectively equality constrained, convex optimization problem, which can then be solved by optimization procedures like the Newton-method [9]. One particularly interesting application of barrier function based MPC formulations is the design of barrier function based continuous-time MPC algorithms, in which the optimal control input is computed as the output of a continuous-time dynamical system, see [3], [5], [10], [11]. These algorithms, which can be seen as system theoretic approach towards algorithmic MPC implementations, are not discussed in detail in this paper, but form our main motivation for the development of smooth barrier function based MPC formulations with guaranteed asymptotic stability.

Let us in the following consider the barrier function based open-loop optimal control problem

\[
\hat{J}^*(x) = \min_u \sum_{k=0}^{N-1} \hat{\ell}(x_k, u_k) + \hat{F}(x_N) \tag{3a}
\]

subject to

\[
\begin{align*}
& x_{k+1} = Ax_k + Bu_k, \quad k = 0, \ldots, N-1 \tag{3b} \\
& x_0 = x := x(k), \tag{3c}
\end{align*}
\]

where \( \hat{\ell}(x,u) := \ell(x,u) + \varepsilon B_u(u) + \varepsilon B_x(x) \) and \( \hat{F}(x) := F(x) + \varepsilon B_f(x) \) are the modified stage and terminal cost terms.

In this section, we will present our main results on asymptotically stabilizing barrier function based linear MPC schemes that rely on polytopic terminal sets of the form
\[
\mathcal{X}_f = \{ x \in \mathbb{R}^n : H_f x \leq 1 \}, \tag{4}
\]

where \( H_f \in \mathbb{R}^{r \times n} \) and \( \mathbb{1} := [1 \cdots 1]^\top \in \mathbb{R}^r \). Note that we can always assume this normalized representation since the terminal set necessarily contains the origin in its interior. As outlined in the introduction, the extension of the existing results on barrier function based linear MPC to such polytopic terminal sets is a natural but nevertheless non-trivial question. Besides being the natural representation of maximal invariant sets for linear systems, polytopic terminal sets might in particular be more suitable to fully exploit the benefits of quadratic barrier function bounds that are valid on a polytopic region in the state space, see [10].
A. Preliminaries

Our goal is to ensure recursive feasibility of the open-loop optimal control problem (3) as well as asymptotic stability of the closed-loop system (1) under the barrier function based MPC feedback \( u(k) = \bar{u}^*_B(x(k)) \) for any feasible initial condition. As in [9] and [10], we will follow the standard approach of using the value function \( J^*(x) \) as a discrete-time Lyapunov function for the closed-loop system. In the following, we introduce some basic definitions and assumptions and summarize important previous results which we will need later in our stability proof.

Definition 1 Let the feasible set \( \mathcal{X}_N \) be defined as \( \mathcal{X}_N := \{ x \in \mathcal{X} : \exists u = \{ u_0, \ldots, u_{N-1} \} s.t. u_k \in \mathcal{U}, k = 0, \ldots, N-1, x_k(u, x) \in \mathcal{X}, k = 1, \ldots, N-1, x_N(u, x) \in \mathcal{X}_f \} \).

Definition 2 In the following, the matrix \( A_K := A+kB \) describes the closed-loop dynamics for a given stabilizing local control law \( u = Kx \) and \( B_K(x) := B_x(x) + B_u(Kx) \) refers to the corresponding combined barrier function of input and state constraints for the set \( \mathcal{X}_K := \{ x \in \mathcal{X} : x = Kx \in \mathcal{U} \} \).

Assumption 1 The state and input constraints are given in form of compact polytopic sets that contain the origin in their interior, i.e., \( \mathcal{X} = \{ x \in \mathbb{R}^n : C_x x \leq d_x \} \) and \( \mathcal{U} = \{ u \in \mathbb{R}^m : C_u u \leq d_u \} \) with \( C_x \in \mathbb{R}^{q_x \times n} \), \( C_u \in \mathbb{R}^{q_u \times m} \) and \( d_x > 0 \), \( d_u > 0 \). Moreover, we assume that the feasible set \( \mathcal{X}_N \) has nonempty interior, i.e., \( \mathcal{X}_N^\circ \neq \emptyset \).

Assumption 2 The barrier functions \( B_u(\cdot) \) and \( B_x(\cdot) \) for the polytopic input and state constraints are gradient centered logarithmic barrier functions of the form \( B_u(u) = \sum_{i=1}^{q_u} B_{u,i}(u) \) and \( B_x(x) = \sum_{i=1}^{q_x} B_{x,i}(x) \), where

\[
B_{x,i}(x) = -\ln(-C_x^i x + d_x^i) + \ln(d_x^i) - \frac{1}{d_x^i} C_x^i x
\]

\[
B_{u,i}(u) = -\ln(-C_u^i u + d_u^i) + \ln(d_u^i) - \frac{1}{d_u^i} C_u^i u
\]

Consequently, the combined barrier function \( B_K(\cdot) \) is given by \( B_K(x) = \sum_{i=1}^{q_x} B_{x,i}(x) + \sum_{i=1}^{q_u} B_{u,i}(Kx) \).

The recentering ensures that \( B_u(0) = 0 \), \( \nabla B_x(x)|_{x=0} = 0 \) and \( B_u(0) = 0 \), \( \nabla B_u(u)|_{u=0} = 0 \), such that the cost function \( J(u, x) \) of the barrier function based problem (3) achieves its (global) minimum at the origin, see [9] and [10] for more details. The following Lemma, which is proven in [10], allows to give a quadratic upper bound for the barrier function \( B_K(\cdot) \) that is valid on a polytopic set.

Lemma 1 ([10]) Let Assumption 1 and Assumption 2 be satisfied. Furthermore, let the matrix \( M \) be defined as \( M := \sum_{i=1}^{q_u} C_u^i C_u^i + K^\top (\sum_{i=1}^{q_x} C_x^i C_x^i) K \) and let \( \gamma \) be a positive scalar with \( \gamma > \frac{1}{2d_{\min}} \), where \( d_{\min} := \min\{d_x^1, \ldots, d_x^{q_x}, d_u^1, \ldots, d_u^{q_u}\} \). Then, it holds that

\[
B_K(x) \leq \gamma x^\top M x \quad \forall x \in \mathcal{P}_K,
\]

where \( \mathcal{P}_K \subset \mathcal{X}_K \) is the polytope

\[
\mathcal{P}_K := \{ x \in \mathbb{R}^n : C_x^i x \leq d_x^i - \frac{1}{2d_{\min}} d_x, \quad i = 1, \ldots, q_x \}
\]

\[
C_u^i K x \leq d_u^i - \frac{1}{2\gamma d_{\min}}, \quad i = 1, \ldots, q_u
\]

Note that, by increasing \( \gamma \), the polytope \( \mathcal{P}_K \) can approximate the set \( \mathcal{X}_K \) in which the local control law is feasible arbitrarily close. Let the terminal cost matrix \( P \succ 0 \) be a solution to the Lyapunov equation

\[
P = A_K^\top P A_K + K^\top R K + Q + \varepsilon \gamma M,
\]

where the matrix \( M \) is defined according to Lemma 1. Under the assumption \( \mathcal{X}_f \subseteq \mathcal{P}_K \) and in combination with the quadratic bound for \( B_K(\cdot) \), this choice of \( P \) allows to compensate the influence of the barrier function \( B_K(\cdot) \) inside the terminal set. The main problem when considering polytopic terminal sets is given by the fact that a polytope, in contrast to an ellipsoid, is defined not only by one inequality but as an intersection of a finite number of halfspaces. This makes it much harder to show a decrease of the terminal set barrier function \( B_f(\cdot) \) under the stabilizing local feedback \( u = Kx \), which is usually needed in order to ensure stability. For example, if we simply considered \( B_f(x) = -\sum_{i=1}^{r} \ln \left( 1 - H_f^i x \right) \), showing a decrease of the function \( B_f(\cdot) \) would be rather difficult as it involves \( r \) different barrier function terms corresponding to the constraints that describe the polytope \( \mathcal{X}_f \). In the following, we first present an approach in which the function \( B_f(\cdot) \) penalizes not all the constraints in \( 1 - H_f x \geq 0 \) but only the worst case constraint, i.e., \( 1 - \max_i \{ H_f^i x \} \). However, with an eye on the usually applied nonlinear programming procedures [4] and considering in particular the outlined barrier function based continuous-time MPC algorithms, we are interested in a smooth formulation of the barrier function \( B_f(\cdot) \). We present therefore in a second step a modified approach, in which the non-smooth maximum operation is approximated by a smooth, convex, and positive definite function.

B. Nonsmooth formulation via the Minkowski functional

We consider in the following the above setup, in which a stabilizing local controller gain \( K \), a bound of the form \( B_K(x) \leq \gamma x^\top M x \forall x \in \mathcal{P}_K \), and a suitable terminal cost matrix \( P \) are given. In order to penalize only the most important constraint of the terminal set description at each time step, we will make use of the Minkowski functional, which is defined in the following and illustrated in Fig. 1.

Definition 3 (C-set, [12]) A C-set is a convex and compact subset of \( \mathbb{R}^n \) that contains the origin as an interior point.

Definition 4 (Minkowski functional, [12]) Let \( S \subset \mathbb{R}^n \) be a given a C-set. Then, the Minkowski functional \( \varphi_S(x) : \mathbb{R}^n \to \mathbb{R}_+ \) is defined as

\[
\varphi_S(x) = \inf \{ \mu \in \mathbb{R}_+ : x \in \mu S \}.
\]

For any C-set \( S \), the Minkowski functional is a continuous, convex, and positive definite function in the variable \( x \).

Assumption 3 Let \( K \in \mathbb{R}^{n \times n} \) be a given stabilizing local controller gain and let the polytope \( \mathcal{P}_K \) be defined according to Lemma 1. We assume the polytopic terminal set \( \mathcal{X}_f := \{ x \in \mathbb{R}^n : H_f x \leq 1 \} \subseteq \mathcal{P}_K \) to be positively invariant under the local system dynamics, i.e., \( x^+ = A_K x \in \mathcal{X}_f \forall x \in \mathcal{X}_f \).
Note that such a set $X_f$ can be constructed easily by scaling some given positively invariant polytopic set $P \subset X_K$ until the condition $X_f = \beta P \subseteq P_K$ is satisfied. A suitable scaling factor is given in the following Lemma, whose proof follows directly from the definition of the polytope $P_K$ and is omitted here for the sake of brevity.

**Lemma 2** Let $\gamma$ and $P_K$ be given according to Lemma 1 and let $P \subset X_K$ be a given polytope. Then, it holds that $\beta P \subseteq P_K$, where the scaling factor $\beta \in (0, 1)$ is given as $\beta = 1 - \frac{d_{\text{min}}}{\gamma}$, with $d_{\text{min}} := \min\{d_1^p, \ldots, d_r^p, d_1^q, \ldots, d_r^q\}$.

Obviously, the Minkowski functional satisfies $\varphi_S(x) = 1 \iff x \in \partial S$ as well as $\varphi_S(x) < 1 \forall x \in S^\circ$. In particular, the Minkowski functional $\varphi_{X_f}(x)$ for the polytopic terminal set $X_f$ is nothing else than $\varphi_{X_f}(x) = \max_i=1,\ldots,r\{H_i^x\}$, i.e., $X_f = \{x \in \mathbb{R}^n : \varphi_{X_f}(x) \leq 1\}$. Based on this, we propose to choose the terminal set barrier function as

$$B_f(x) = -\ln(1 - \varphi_{X_f}(x)).$$

(9)

Clearly, $B_f : X_f^o \to \mathbb{R}_+$ is a positive definite and convex function which attains its minimum at the origin and satisfies $B_f(x) \to \infty$ whenever $x \to \partial X_f$. We can now state the following result concerning the feasibility and stability properties of the closed-loop system.

**Theorem 1** Let the Assumptions 1, 2, and 3 be satisfied and let the terminal cost matrix $P$ be given by (7). Furthermore, let the terminal set barrier function $B_f(\cdot)$ be given by (9). Then, the barrier function based MPC feedback $u(k) = \hat{u}_0(x(k))$ resulting from problem (3) asymptotically stabilizes the closed-loop system (1) under strict satisfaction of all input and state constraints for any initial condition $x(0) \in X_N^o$.

**Proof:** We first show recursive feasibility of problem (3) based on standard arguments, cf. [1]. For any $x_0 \in X_N^o$, there exists by definition an optimal input sequence $\hat{u}^*(x_0) = \{\hat{u}_0, \ldots, \hat{u}_{N-1}\}$ that guarantees strict satisfaction of all input, state, and terminal set constraints and results in the feasible open-loop state sequence $x^*(0) = \{x_0, x_1^0(x_0), \ldots, x_N^0(x_0)\}$ with $x_N^0 := x_N^0(x_0) \in X_f^o$. The successor state $x_{N+1}^0$ of the feedback is given as $x_{N+1}^0 = x_1^0(x_0) = A x_0 + B \hat{u}_0^T$. Then, due to the properties of the terminal set $X_f$, see Assumption 3, $\hat{u}(x_0) = \{\hat{u}_0^*, \ldots, \hat{u}_{N-1}^*, AK x_N^0\}$ is a suboptimal but feasible input sequence for the initial state $x_0^m$ that results in the in the feasible open-loop state sequence $x^m(0) = \{x_0^m, x_1^m(x_0), \ldots, x_N^m(x_0), A x_N^m(x_0)\}$ with $A x_N^m(x_0) \in X_f^o$. This shows that for any $x_0 \in X_N^o$, there exists a feasible input sequence that ensures that the successor state $x_{N+1} = A x_0 + B \hat{u}_0^T$ lies again in the interior of the feasible set $X_f$, which guarantees recursive feasibility of the open-loop optimal control problem (3).

In the following, we show that the value function satisfies

$$J^*(x_N^0) - J^*(x_0) \leq -\ell(x_0, \hat{u}_0(x_0)) \forall x_0 \in X_N^o,$$

(10)

which can then be used to prove asymptotic stability. First, due to suboptimality of the input sequence $\hat{u}(x_0)$, it holds that $\hat{J}^*(x_N^0) - \hat{J}^*(x_0) \leq \hat{J}(\hat{u}(x_0), x_N^0) - \hat{J}^*(x_0)$. Here, $\hat{J}(\hat{u}(x_0), x_N^0)$ denotes the value of the cost function evaluated for the suboptimal input sequence $\hat{u}(x_0)$. Moreover, $\hat{J}(\hat{u}(x_0), x_N^0) - \hat{J}^*(x_0) = \hat{F}(AK x_N^0) - \hat{F}(x_N^0) + \ell(x_N^0, K x_N^0, -\ell(x_0, \hat{u}_0))$ since

$$\hat{F}(AK x_N^0) - \hat{F}(x_N^0) + \ell(x_N^0, K x_N^0)$$

(11a)

$$= ||A x_N^0||_F^2 - ||x_N||_F^2 + ||x_N||_F^2 - ||K x_N^0||_R^2$$

(11b)

$$+ \varepsilon_B K x_N^0 - \varepsilon_B x_N^0(\hat{F}(x_N^0) - \hat{F}(x_N^0)) \leq \varepsilon_B (AK x_N^0 - \hat{F}(x_N^0)) \leq 0 \forall x_N^0 \in X_f^o. \quad (11c)$$

Here, the first inequality follows from the quadratic bound $B_f(k) \leq \varepsilon_B^t M x_0 \forall x_0 \in X_f \subseteq P_K$, see Lemma 1, and the choice of the terminal cost matrix $P$ given in (7). Furthermore, let $\varphi_{X_f}(x_N^0) := \mu$, i.e., $x_N^0 \in \mu X_f$ with $\mu < 1$ for any $x_N^0 \in X_f^o$. Then, due to the invariance of the polytopic terminal set, we know that $A x_N^0 \in \mu X_f^o$, and, hence, $\varphi_{X_f}(A x_N^0) \leq \mu = \varphi_{X_f}(x_N^0) \forall x_N^0 \in X_f^o$. Since the natural logarithm is a monotone function, this directly implies that $B_f(A x_N^0) - B_f(x_N^0) \leq 0 \forall x_N^0 \in X_f^o$. Finally, the well-defined problem setup and the construction of the involved barrier functions, see Assumptions 1 and 2, ensure that $J^*(x)$ is a well-defined and positive definite function with $J^*(x) \to \infty$ whenever $x \to \partial X_f$. Thus, in combination with the decrease property (10), $J^*(x)$ can be used as a Lyapunov function which proves asymptotic stability of the origin of system (1) under the feedback $u(k) = \hat{u}(x(k))$ for any $x(0) \in X_N^o$. □

C. Smooth formulation based on contractive terminal sets and a smooth approximation of the Minkowski functional

While the approach presented in the previous section in principle allows to design an asymptotically stabilizing barrier function based MPC scheme, the usage of the nonsmooth Minkowski functional makes it not really suitable for the application of conventional smooth nonlinear programming algorithms. In the following, we present a modified approach for the incorporation of polytopic terminal sets which is based on a continuously differentiable approximation of the Minkowski functional and thus allows a smooth formulation of the underlying terminal set barrier function. In order to guarantee asymptotic stability of the closed-loop system, we require that the polytopic terminal set is not only positively invariant but in fact $\lambda$-contractive with respect to the given stabilizing local control law.
Definition 5 (λ-contractive set) Given a positive scalar \( \lambda \in (0, 1) \), a C-set \( S \) is called \( \lambda \)-contractive for a linear discrete-time system of the form \( x^+ = Ax \) if it satisfies \( x^+ = Ax \in \lambda S \) for all \( x \in S \).

Assumption 4 Let \( K \subset \mathbb{R}^m \times \mathbb{R}^n \) be a given stabilizing local controller gain and let the polytope \( P_K \) be defined according to Lemma 1. We assume the polytopic terminal set \( X_f := \{ x \in \mathbb{R}^n : H_f x \leq 1 \} \subset P_K \) to be \( \lambda \)-contractive for the local system dynamics \( x^+ = A_K x \) for a given \( \lambda \in (0, 1) \).

Remark 1 Let \( \{v_1, \ldots, v_n\} \) be the vertices of the \( \lambda \)-contractive polytopic set \( X_f \) for a given \( \lambda \in (0, 1) \). Then, due to the convexity of the set \( X_f \), it holds that \( \lambda X_f \subset X_f \) if and only if \( \varphi_{X_f}(v_j) = \lambda \varphi_{X_f}(v_j) < 1 \) for \( j = 1, \ldots, n \).

Remark 2 Let \( \{P_1, \ldots, P_N\} \) be a partition of the set \( X_f \). Then, the terminal set condition is satisfied.

Lemma 4 Let the terminal set \( X_f \) be given according to Assumption 4 and let its smooth approximation \( X_f^p \) be defined by (14). Then, for any \( \lambda \in (0, 1) \), there exists an integer \( p_\lambda \in [1, \infty) \) such that \( \lambda X_f \subset X_f^p \) for any \( p \geq p_\lambda \).

Proof: As outlined above, \( X_f^p \subset X_f \) is always satisfied by construction as \( \varphi_{X_f}(x) > \varphi_{X_f}(x) \) \( \forall x \in \mathbb{R}^n \) for any \( p \geq 1 \). Furthermore, by Lemma 3, the set \( X_f^p \) uniformly converges to \( X_f \) from the inside as \( p \to \infty \). Hence, if \( \lambda \in (0, 1) \), we can always find a finite \( p_\lambda \) such that \( \lambda X_f \subset X_f^p \).

A graphical illustration of the sets \( X_f, \lambda X_f, \) and \( X_f^p \) is given in Fig. 1. Based on the above results, we now propose to use the terminal set barrier function

\[
B_f(x) = -\ln \left( 1 - \left( \varphi_{X_f}(x) \right)^p \right),
\]

which is by construction twice continuously differentiable, positive definite, and convex on its domain \( X_f^p \). This allows to formulate the overall barrier function based open-loop optimal control problem (3) as an unconstrained minimization problem with a smooth and strongly convex cost function. In addition, we can give the following results on stability properties of the closed-loop system.

Theorem 2 Let the Assumptions 1, 2, and 4 hold true and let the terminal cost matrix \( P \) be given by (7). Furthermore, let the terminal set barrier function \( B_f(\cdot) \) be chosen according to (15), where the integer parameter \( p \) is assumed to ensure \( \lambda X_f \subset X_f^p \subset X_f \), see Lemma 4. Let \( X_f^p \) denote the feasible set related to the approximated terminal set \( X_f^p \) in (14), see Definition 1. Then, the barrier function based MPC feedback \( u(k) = \hat{u}_p(x(k)) \) resulting from problem (3)

- asymptotically stabilizes the closed-loop system (1) under strict satisfaction of all state and input constraints for any initial condition \( x(0) \in X_f^0 \).

Proof: The proof is similar to the proof of Theorem 1. We know that for any \( x_0 \in X_f^0 \) there exists an optimal input sequence \( \hat{u}^*(0) = \{\hat{u}_0^*, \ldots, \hat{u}_{N-1}^*\} \) that ensures strict satisfaction of all state, input, and terminal set constraints, i.e., in particular, \( x_N^* = x_N(\hat{x}_0) \in X_f^p \). Due to the convexity of the terminal set and \( p \geq p_\lambda \), it holds that \( A_K x_N^* \in \lambda X_f^p \subset X_f^p \) for any \( x_N^* \in X_f^p \). Hence, \( \hat{u}(x_0) = \{\hat{u}_0^*, \ldots, \hat{u}_{N-1}^*, K x_N^*\} \) is a suboptimal but feasible input sequence for the successor state \( x_N^* = Ax_0 + Bu_0^* \), which ensures recursive feasibility of the open-loop optimal control problem (3).

Furthermore, for any \( x_N^* \in X_f^p \), we can define \( \mu_1 := \varphi_{X_f}(x_N^*) < 1 \) and \( \mu_2 := \varphi_{X_f}(x_N^*) \leq \mu_1 \), i.e., \( x_N^* \in \mu_2 X_f^p \). By the contructivity of the set \( X_f \), we know that \( A_K x_N^* = \lambda \mu_2 x_N^* \), which means that we can use the representation \( A_K x_N^* = \lambda \mu_2 \sum_{j=1}^{n_N} v_j \theta_j \) with \( \theta_j \geq 0 \), \( \sum_{j=1}^{n_N} \theta_j \leq 1 \),
where \(v_1, \ldots, v_{n_v}\) are the vertices of the terminal set \(X_f\). With this, we get for any \(x^*_N \in X_f^\circ\)
\[
\varphi_{X_f}^p(A_K x_N) = \varphi_{X_f}^p \left( \lambda \mu_2 \sum_{j=1}^{n_v} \theta_j v_j \right)
\]
\[
\leq \mu_2 \sum_{j=1}^{n_v} \theta_j \varphi_{X_f}^p(v_j) < \mu_2 \sum_{j=1}^{n_v} \theta_j \leq \mu_2 ,
\]
where we used the convexity of the function \(\varphi_{X_f}^p(.)\) and the fact that \(p \geq p_\lambda\), i.e. \(\lambda X_f \subset X_f^p\), see Remark 1. Hence, we have \(\varphi_{X_f}^p(A_K x_N) < \mu_2 < \mu_1 := \varphi_{X_f}^p(x_N^\ast)\), which directly yields \(B_f(A_K x_N^\ast) - B_f(x_N^\ast) \leq 0 \forall x_N^\ast \in X_f^p\). Due to \(X_f^p \subset X_f \subseteq \mathcal{P}_K\) and the choice of \(P\), this condition is sufficient for ensuring the decrease property \(J^*(x_0^\ast) - J^*(x_0) \leq -\ell(x_0, u_0(x_0)) \forall x_0 \in X_N^p\), cf. the proof of Theorem 1.

This shows that we can again use the value function \(J^*(x)\) as a Lyapunov function of the closed-loop system, which allows to prove asymptotic stability of the origin. \(\square\)

**Remark 2** It can be shown that the resulting feasible set satisfies \(\lambda X_N^\ast \subset X_N^p \subset X_N^\ast\) for any \(\lambda \in (0,1)\) and any \(p \geq p_\lambda\), which illustrates that the results from the nonsmooth formulation may be recovered for \(\lambda \to 1, p \to \infty\).

**IV. Numerical Example**

We briefly illustrate the closed-loop behavior of the proposed MPC algorithm by means of an academic numerical example. We consider a double integrator system with the discrete-time system model
\[
x(k+1) = \begin{bmatrix} 1 & T_s \\ 0 & 1 \end{bmatrix} x(k) + \begin{bmatrix} T_s^2 \\ T_s \end{bmatrix} u(k),
\]
where the discretization time is chosen to be \(T_s = 0.1\) s. The linear MPC open-loop optimal control problem is formulated for \(N = 10\), \(Q = \text{diag}(1, 0.1)\), \(R = 1\) and the asymmetric input and state constraints \(-2 \leq u \leq 1, -2 \leq x_1 \leq 3, \) and \(-0.8 \leq x_2 \leq 0.6\). The parameters of the barrier function based MPC formulation are chosen according to the design procedure presented in Section III with \(\varepsilon = 10^{-2}\) and \(\gamma = 100\). The \(\lambda\)-contractive terminal set is computed for \(\lambda = 0.99\), which results in \(p = p_\lambda = 56\). Exemplary simulation results are illustrated in Fig. 2 together with some comments. It has to be noted that the proposed design procedure results in considerably enlarged terminal and feasible sets. In all simulations, the numerical performance was comparable to that of existing methods based on ellipsoidal terminal sets.

**V. Conclusion**

In this paper we presented a novel approach for the design of barrier function based linear model predictive control schemes that make use of polytopic terminal sets. We showed how the corresponding cost function can be formulated in a smooth way and proved nominal asymptotic stability of the closed-loop system. Moreover, we illustrated that in the presence of asymmetric input and state constraints the use of polytopic terminal sets may result in a considerably enlarged region of attraction.

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**REFERENCES**


