Stabilization of LPV Positive Systems

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Abstract—This paper considers the stabilization issue for continuous-time linear parameter varying (LPV) positive systems. The time varying parameters are known and are modeled as belonging to the simplex set. The proposed stabilization approach relies on a parameter dependent Lyapunov function combined with a subtle choice of a slack variable that is not necessary diagonal. In fact, due to the positivity constraint on the closed-loop system the slack variable is chosen to be a Metzler matrix. Indeed, the particular case when the slack matrix is diagonal may work but the resulting stabilization conditions can be conservative. This fact is illustrated by a comparison example.

I. INTRODUCTION

The main objective of the paper is to provide a flexible control design approach for continuous-time linear parameter varying (LPV) positive systems. A system is referred to be positive if it involves states that only take nonnegative values (see [9], [25], [16], [21] for general references). Such kind of system for which the states take physical nonnegative values, can be encountered in a wide range of applications such as in industrial processes involving chemical reactors, in water storage systems and in atmospheric pollution models. Other variety of systems exhibiting the positivity constraint on their states can be found in management science, economics, biology, medicine, etc.

General LPV systems have been treated extensively in the literature. For a comprehensive overview on LPV systems see the survey paper [27]. To the best of our knowledge, LPV positive systems are not yet treated in the literature. Only positive systems with parametric uncertainties have been considered in [1], [15]. In contrast, LTI and switched positive systems are well-reported and well-developed in the literature, without been exhaustive, see for instance, [22], [2], [3], [29], [26], [5], [12], [7], [19], [18], [11] in stability and control context and, [20], [4], [24], [6] in estimation context.

This paper treats the synthesis of gain scheduled control laws that stabilize and maintain the positivity of a given continuous-time LPV positive system for which the time varying parameter are assumed to be known

\[ \dot{x}(t) = A(\theta)x(t) \]  

(1a)

and are modeled as belonging to the simplex set. For this purpose, two LMI-based procedures are provided. Also, an explanation on how to incorporate extra constraints on the gain scheduled control laws is presented. The proposed approach is based on a parameter dependent Lyapunov function combined with decoupling technique between the Lyapunov variables and the controller variables. This procedure leads to easy computational conditions. The decoupling technique has been proved to be efficient and has been used in different contexts [23], [13], [28], [14], [17]. This technique uses extra slack variables which leads to less conservative synthesis conditions. Due to the positivity constraint on the closed-loop system, the synthesis problem is not possible without a suitable choice of a special slack variable. Of course, one can use a slack variable as a diagonal matrix. However, this choice leads to conservative stabilization conditions since it provide a few degree of freedoms. Instead, we propose a subtle choice of slack variables that are Metzler matrices. Our numerical results show the highly advantage of the use of such slack variables.

This paper is organized as follows. The description and characterization of positive LPV systems is presented in Section 2. Section 3 is devoted to the synthesis of gain scheduled control laws that stabilize and maintain the positivity of a given LPV system. Two numerical control designs are given. Also, it is shown how to add extra constraints on the gain scheduled control laws. In section 4, a numerical comparison of the proposed numerical procedures is provided. Concluding remarks are presented in the last section.

Notations: \( M' \) denotes the transpose of the real matrix \( M \). For a real matrix \( M \) (resp. a vector \( v \)), \( M \geq 0 \) (resp. \( v \geq 0 \)) means that its components are nonnegative: \( M_{ij} \geq 0 \) (resp. \( v_i \geq 0 \)). \( M \leq \) means \(-M \geq 0 \). For symmetric real matrix \( M \), \( M > 0 \) (resp. \( M < 0 \)), means that \( M \) is definite positive (resp. negative).

II. POSITIVE LPV SYSTEMS

Here, our main objective is to recall some facts on positive systems and to provide a relatively concise characterization of an LPV positive system described by

\[ \dot{x}(t) = A(\theta)x(t) \]

with \( A(\theta) = \sum_{j=1}^{p} \theta_j A_j \), where \( x \in \mathbb{R}^n \) is the state vector and \( \theta \) is an exogenous parameter that can be time dependent.
such that
\[ \sum_{j=1}^{p} \theta_j(t) = 1, \; \theta_j(t) \geq 0, \; \forall t \geq 0, \; j = 1, \ldots, p. \] (2)

Throughout this paper the free system is assumed to satisfy a positivity constraint on its states as follows.

**Definition 1:** System (1) is said to be positive if for any nonnegative initial conditions \( x(0) \geq 0 \), the corresponding trajectory is nonnegative: \( x(t) \geq 0 \) for all \( t \geq 0 \).

An inherent property from the positivity of system (1) is related to Metzlerian matrices.

**Definition 2:** A matrix \( M \) is called a Metzler matrix if its off-diagonal elements are nonnegative: \( M_{ij} \geq 0, \; i \neq j \).

**Remark 1:** \( M \) is Metzler matrix is equivalent to the fact that there exists a diagonal matrix \( D \) such that \( M + D \) is positive: \( M + D \geq 0 \).

**Definition 3:** A real matrix \( M \) is called a positive matrix if all its elements are nonnegative: \( M_{ij} \geq 0 \). Also, \( M \) is called negative matrix if \( -M \) is positive.

For a more complete treatment for Metzler and positive matrices and their properties see e.g. [10], [8] and [16]. The following result provides an useful property of Metzler matrices that are Hurwitz (see for instance [10]).

**Lemma 1:** Let \( M \) be a Metzler matrix, then the following statements are equivalent

(i) \( M \) is Hurwitz.

(ii) \( M^{-1} \leq 0 \).

The following result can be viewed as an extension of the classical positivity result for LTI systems [25].

**Lemma 2:** System (1) is positive for any parameter function \( \theta \) taking values in the simplex set, if and only if the matrices \( A_1, \ldots, A_p \) are Metzler matrices.

Now, consider the following dual system
\[
\dot{z}(t) = A'(\theta)z(t)
\] (3)

with \( A'(\theta) = \sum_{j=1}^{p} \theta_j A'_j \) and which depends on the same exogenous parameter vector \( \theta \) as for system (1).

In the sequel, we will exploit the two common properties connected to positivity and stability of system (1) and its dual. In fact, by the result of Lemma 2 we have that system (1) is positive if and only if its dual system (3) is positive. Also, by duality we have that system (1) is asymptotically stable if and only if its dual system (3) is asymptotically stable.

### III. Control Design

In this section, we treat the synthesis problem for LPV positive systems by means of gain scheduled control laws. The proposed approach is based on a parameter-dependent Lyapunov function combined with a suitable choice of slack matrix variables. In fact, it will be shown that Metzler matrix slack variables play a key role in ensuring positivity and stability of the closed-loop system. In addition, they can be useful for taking into account additional constraints on the stabilizing control laws.

Consider the following LPV system
\[
\dot{x} = A(\theta)x + B(\theta)u
\] (4)

with \( A(\theta) = \sum_{j=1}^{p} \theta_j A_j \) and \( B(\theta) = \sum_{j=1}^{p} \theta_j B_j \), where \( x \in \mathbb{R}^n \) is the state vector, \( u \in \mathbb{R}^m \) is the control input vector and \( \theta \) is an exogenous parameter that can be time dependent with bounded derivatives such as
\[
\sum_{i=1}^{p} \theta_i(t) = 1, \; \theta_i(t) \geq 0, \; \forall t \geq 0, \; i = 1, \ldots, p \] (5)

\[
\theta_i(t) \leq \beta_i, \; \forall t \geq 0, \; i = 1, \ldots, p \] (6)

where \( \beta_i \) are assumed to be known bounds on the derivatives of the parameters \( \theta_i \).

Our objective is to design a gain scheduling control law of the form
\[
u(t) = \sum_{j=1}^{p} \theta_j K_j x
\] (7)

that stabilizes system (4) and maintains the positivity of the closed-loop system. For this purpose, we provide the following result.

**Theorem 1:** There exists a state-feedback of the form (7) that stabilizes and maintains the positivity of system (4), if there exist matrices \( P_1, \ldots, P_p, Y_1, \ldots, Y_p \), a diagonal matrix \( D \) and a scalar \( \alpha \) such that the following conditions hold:
\[
M_{ii} < 0, \; P_i > 0 \quad \text{for} \quad i = 1, \ldots, p
\] (8a)
\[
M_{ij} + M_{ji} < 0, \quad \text{for} \quad i < j = 1, \ldots, p
\] (8b)
\[
A_i D + B_i Y_j + \alpha I \leq 0, \quad \text{for} \quad i, j = 1, \ldots, p
\] (8c)

with
\[
M_{ij} = \begin{bmatrix}
\sum_{k=1}^{p} \beta_k P_k - DA'_i - A_i D - Y'_j B_i - B_i Y_j & P_i + D - A_i D - B_i Y_j \\
P_i + D - DA'_j - Y'_j B_i & 2D
\end{bmatrix}
\]
If so, a stabilizing gain $K(\theta) = \sum_{j=1}^{p} \theta_j K_j$ that maintains the closed-loop system positive is given by

$$K_i = Y_i D^{-1}, \quad i = 1, \ldots, p.$$  

**Proof:** The proof is based on duality. We shall exploit the fact that with the same control law, the positivity and the stability of system (4) is equivalent to the positivity and the stability of the dual closed-loop system given by

$$\dot{z} = A_{cl}^{'}(\theta)z$$  

(9)

where $A_{cl} = A(\theta) + B(\theta)K(\theta)$, with $K(\theta) = \sum_{j=1}^{p} \theta_j K_j$.

Now, we shall first prove the stability of the dual system (9) by considering the following parameter-dependent Lyapunov function:

$$V = z^{'}P(\theta)z$$  

(10)

where $P(\theta) = \sum_{j=1}^{p} \theta_j P_j$. Note that since all $P_j$ are positive definite matrices we have that $P(\theta) > 0$.

The time derivative of $V$ is given by:

$$\dot{V} = z^{'}P_\theta z + z^{'}P(\theta)z + z^{'}P(\theta)z$$  

(11)

with $P_\theta = \sum_{j=1}^{p} \theta_j P_j$.

In order to decouple the Lyapunov matrices from the system matrices, slack matrix variables are introduced as follows:

$$2[z^{'}D + z^{'}D][\dot{z} - A_{cl}^{'}(\theta)z] = 0$$  

(12)

where $D$ is a diagonal matrix.

From (11) and (12), it follows that:

$$\dot{V} = z^{'}P_\theta z + z^{'}P(\theta)z + z^{'}P(\theta)z + 2[z^{'}D + z^{'}D][\dot{z} - A_{cl}^{'}(\theta)z]$$  

(13)

Since $\hat{\theta}_j \leq \beta_j$, $j = 1, \ldots, p$, then $V$ satisfies:

$$\dot{V} \leq z^{'}(\sum_{j=1}^{p} \beta_j P_j)z + z^{'}P(\theta)z + z^{'}P(\theta)z + 2[z^{'}D + z^{'}D][\dot{z} - A_{cl}^{'}(\theta)z]$$  

(14)

By substituting $A_{cl}$ by its expression, we obtain:

$$\dot{V} \leq z^{'}(\sum_{j=1}^{p} \beta_j P_j)z + z^{'}P(\theta)z + z^{'}P(\theta)z + 2[z^{'}D + z^{'}D][\dot{z} - (A(\theta) + B(\theta)K(\theta))^{'\theta}z]$$  

(15)

By substituting $A(\theta)$, $B(\theta)$, $K(\theta)$ and $P(\theta)$ by their expressions and taking $\dot{\xi} = \begin{bmatrix} z \\ \dot{z} \end{bmatrix}$, we have

$$\dot{V} \leq \sum_{j=1}^{p} \sum_{i=1}^{p} \theta_j \theta_i M_{ij} \xi$$  

(16)

where

$$M_{ij} = \begin{bmatrix} \sum_{i=1}^{p} \beta_i P_i - D\beta_i' - A_i D - Y_i B_i' - B_i Y_i & P_i + D - A_i D - B_i Y_i \\ P_i + D - D\beta_i' - Y_i B_i' & 2D \end{bmatrix}$$

with $Y_i = K_i D$.

(16) can be rewritten as

$$\dot{V} \leq \sum_{i=1}^{p} \sum_{j=1}^{p} \theta_i \theta_j \xi M_{ij} \xi + \sum_{i < j} \theta_i \theta_j \xi(M_{ij} + M_{ji}) \xi$$  

(17)

The conditions of the theorem ensure that $\dot{V} < 0$ which is implied by:

$$M_{ii} < 0, \quad M_{ij} + M_{ji} < 0, \quad i < j = 1, \ldots, p$$  

(18)

$$M_{ii} < 0, \quad M_{ij} + M_{ji} < 0, \quad i < j = 1, \ldots, p$$  

(19)

In order to show that the closed-loop system is positive, it suffices to ensure that each of its modes is a Metzler matrix. That is $A_i + B_i K_j$ is Metzler for all $i, j$. Since $D < 0$ and $D$ is diagonal, we obtain from condition (8c): $A_i D + B_i Y_j + \alpha I \leq 0$, that

$$(A_i D + B_i Y_j + \alpha I) D^{-1} \geq 0.$$  

Thus, we have $A_i + B_i K_j + \alpha I D^{-1} \geq 0$ which means by Remark 1 that $A_i + B_i K_j$ is a Metzler matrix.

Note that the previous result involves a diagonal slack variable which provides a few degrees of freedom. A less conservative stabilization conditions can be established by using a Metzler matrix as a slack variable instead of a diagonal one. This is stated in the following result.

**Theorem 2:** Let $\gamma > 0$ be a given scalar. There exist a state-feedback that stabilizes and maintains the positivity of system (4), if there exist matrices $P_1, \ldots, P_p$, $Y_1, \ldots, Y_p$, a matrix $G$ and a positive scalar $\alpha$ such that the following conditions hold

$$M_{ii} < 0, \quad P_i > 0 \quad \text{for} \quad i = 1, \ldots, p$$  

(20a)

$$M_{ij} + M_{ji} < 0, \quad \text{for} \quad i < j = 1, \ldots, p$$  

(20b)

$$A_i G' + B_i Y_j + \gamma G' \leq 0, \quad \text{for} \quad i < j = 1, \ldots, p$$  

(20c)

$$G + \alpha I \geq 0$$  

(20d)

with

$$Z_{ij} = \begin{bmatrix} \sum_{j=1}^{p} \beta_i P_j - GA_i' - A_i G' - Y_i B_i' - B_i Y_i & P_i + G' - A_i G' - B_i Y_i \\ P_i + G' - G A_i' - Y_i B_i' & G + G' \end{bmatrix}$$

If so, a stabilizing gain $K(\theta) = \sum_{j=1}^{p} \theta_j K_j$ that maintains the closed-loop system positive is given by

$$K_i = Y_i G^{-1}, \quad i = 1, \ldots, p.$$  

**Proof:** By following the same line of arguments in the previous result, we have that $V = \dot{z}^{'}P(\theta)z$ with
\(P(\theta) = \sum_{j=1}^{p} \theta_j P_j\) is a Lyapunov function. Thus, the state-feedback control \(u = \sum_{j=1}^{p} \theta_j K_j x\) with \(K_i = Y_i(G')^{-1}\) is stabilizing. In order to complete the proof, we only need to show that the closed-loop system is positive or equivalently that each of its modes is a Metzler matrix.

Let \(G\) be any matrix satisfying condition (20a) and (20b) which implies that the block matrix \(G + G'\) satisfies \(G + G' < 0\). In addition, conditions (20d) is equivalent to the fact that \(G\) is a Metzler matrix. Then, according to Lemma 1 the inverse of \(G\) is a negative matrix \((G^{-1} \leq 0)\). Due to this fact, we have that

\[(A_i G' + B_i Y_j + \gamma G')(G')^{-1} \geq 0, \quad \text{for } i, j = 1, \ldots, p.\]

Since by condition (20c), the matrix \(A_i G' + BY_j + \gamma G'\) is negative. Hence, it holds for each mode

\[(A_i G' + B_i Y_j + \gamma G')(G')^{-1} = A_i + B_i K_j + \gamma I \geq 0,\]

which in turn shows that every mode of the closed-loop system is a Metzler matrix. Then, by Lemma 2 we can conclude that the state-feedback control \(u = \sum_{j=1}^{p} \theta_j K_j x\) with \(K_i = Y_i(G')^{-1}\) maintains the positivity of the closed-loop system and the proof is complete.

Our approach is flexible due to the fact that it can incorporate constraints on the stabilizing control laws. For instance, if a control law is needed to be nonnegative: \(u(t) \geq 0, \forall t \geq 0\), such control can be computed by combining conditions (20a)-(20d) with the additional constraints:

\[Y_i \leq 0, \quad i = 1, \ldots, p.\]

In fact, due to the nonnegativity of the states, the nonnegativity of a control law of the form \(u = \sum_{j=1}^{p} \theta_j K_j x\) is insured by the fact that the gains \(K_i\) must be positive matrices: \(K_i \geq 0\). Indeed, this condition can be recovered from

\[Y_i \leq 0 \Rightarrow Y_i(G')^{-1} = K_i \geq 0,\]

as we have previously shown that we can use a matrix \(G\) with negative inverse.

In the same spirit, one can take into account prescribed upper and lower bounds on the gains of the control law such as: \(\underline{K} \leq K_i \leq \overline{K}\). For this purpose, it suffices to add to conditions (20a)-(20d) the following extra constraints:

\[\overline{K} G' \leq Y_i \leq \underline{K} G', \quad i = 1, \ldots, p.\]

IV. Numerical results

Consider system (1) involving constant parameters \(a\) and \(b\) such that

\[
A_1 = \begin{bmatrix} 0 & 3 \\ a + 1 & -1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 1 \\ b + 3 & -1 \end{bmatrix} \quad (21)
\]

\[
B_1 = \begin{bmatrix} 2a \\ -1 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 2 - b \\ 1 \end{bmatrix} \quad (22)
\]

Based on Lemma 2, one can easily see that the free system is positive in the range: \(-1 \leq a\) and \(-3 \leq b\) since with these values \(A_1\) and \(A_2\) are Metzler matrices.

The stabilizable region of the proposed approaches was investigated for various values of \(a\) and \(b\). Figure 1 shows the stabilizable region (in blue color) of the parameter-dependent Lyapunov function approach with \(G\) diagonal for \(\beta_j = 0.99, j = 1, \ldots, 2\). The stabilization result is significantly improved by using a general Metzler matrix variable \(G\) instead of a diagonal matrix as illustrated in figure 2. We can see that the stabilizable region is expanded by using Theorem 2.

![Fig. 1. The stabilizable region of the parameter-dependent Lyapunov function approach with \(G\) diagonal for \(\beta_j = 0.99, j = 1, \ldots, 2\).](image1)

![Fig. 2. The stabilizable region of the parameter-dependent Lyapunov function approach with a Metzler \(G\) for \(\beta_j = 0.99, j = 1, \ldots, 2\).](image2)
V. CONCLUSIONS

We have treated the stabilization problem for time-varying continuous-time positive systems by using a new Lyapunov-based technique that involves Metzler slack variables. Due to the positivity constraint on the closed-loop system this kind of slack variable are suitable. This choice of slack variables leads to an adequate stabilization result which can outperform the case when diagonal slack variables are involved. Our approach has been illustrated by a comparison example.

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