The disturbance decoupling problem for continuous piecewise affine systems

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Abstract—In this paper we study the disturbance decoupling problem for continuous piecewise affine systems. We establish a set of necessary conditions and a set of sufficient conditions, both geometric in nature, for such systems to be disturbance decoupled. Furthermore, we investigate mode-independent state feedback controllers for piecewise affine systems and provide sufficient conditions for the solvability of the disturbance decoupling problem by state feedback.

I. INTRODUCTION

The disturbance decoupling problem amounts to finding a feedback law that eliminates the effect of disturbances on the output of a given input/state/output dynamical system. The investigation of this problem has been the starting point for the development of geometric control theory [1], [2], [12]. For both linear and (smooth) nonlinear systems, geometric control theory has been proven to be very efficient in solving various control problems, including the disturbance decoupling problem (see e.g. [3]–[5], [10], [11]).

In the context of hybrid dynamical systems, the results on disturbance decoupling are limited to switched linear systems [6], [13]. In this paper, we study a particular class of hybrid systems, namely continuous piecewise affine systems.

The major difference between switched linear systems and piecewise affine systems is the nature of the switching behavior. For piecewise affine systems the switching behavior is state-dependent whereas it is state-independent for switched linear systems.

For state-independent switching case, the solution of the disturbance decoupling problem can be obtained by following mainly the footsteps of the (non-switching) linear case. Indeed, a noteworthy consequence of the state-independent switching is that the set of reachable states under the influence of disturbances is a subspace. This allows one to generalize the so-called controlled invariant subspaces of linear systems to switched linear systems. Such a generalization leads to elegant necessary and sufficient conditions [6], [13] for a switched linear system to be disturbance decoupled. In the same papers, disturbance decoupling problems by different feedback schemes have also been solved based on these necessary and sufficient conditions.

However, a similar approach breaks down in the case of state-dependent switching as the set of reachable states under the influence of disturbances is not anymore a subspace, not even a convex set in general. As such, neither the results nor the approach adopted for the state-independent case can be applied to state-dependent switching case.

In this paper, we develop a new approach that takes into account the state-dependent switching behavior of piecewise affine systems. This approach allows us to provide a set of necessary conditions and a set of sufficient conditions under which a continuous piecewise affine system is disturbance decoupled. Although these conditions do not coincide in general, we point out some special cases in which they do coincide. Furthermore, we present conditions for the existence of mode-independent static feedback controllers that render the closed-loop system disturbance decoupled. All the conditions we present are geometric in nature and easily verifiable.

The structure of the paper is as follows. In Section II we introduce the class of continuous piecewise affine systems. For this class of systems, we define the disturbance decoupling problem in Section III and give a set of necessary conditions and a set of sufficient conditions for such a system to be disturbance decoupled. In Section IV, we provide conditions under which the necessary conditions and the sufficient conditions coincide. The problem of disturbance decoupling by state feedback is discussed in Section V. Finally, Section VI contains the main conclusions of the paper.

Throughout the paper we use the following notational conventions and concepts from geometric control theory (see e.g. [10] for more details). Let $A \in \mathbb{R}^{n \times n}$ and $V$ be a subspaces of $\mathbb{R}^n$. We say that $V$ is $A$-invariant if $AV \subseteq V$. Let $B \in \mathbb{R}^{n \times m}$. With $\langle A \ | \ im \ B \rangle$, we denote the subspace $im \ B + im \ AB + \cdots + im \ A^{m-1} B$, which is the smallest $A$-invariant subspace containing $im \ B$.

II. PIECEWISE AFFINE SYSTEMS

Before we can define the class of continuous piecewise affine systems, we need the notions of affine functions and piecewise affine functions. An affine function is a function $\theta : \mathbb{R}^m \to \mathbb{R}^n$ of the form $\theta(x) = Qx + q$, with $Q$ a $m \times n$ matrix and $q$ an $m$-vector. A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is called piecewise affine if there exists a finite set of affine functions $f_i : \mathbb{R}^n \to \mathbb{R}^{m_i}$, $i = 1, 2, \ldots, p$, such that

$$f(x) \in \{f_1(x), f_2(x), \ldots, f_p(x)\}$$

for all $x \in \mathbb{R}^n$. The domain of a continuous piecewise affine function can be divided into a set of polyhedral regions in such a way that the restriction of the function $f$ to any of the regions is given by an affine function [7, Prop. 2.2.3]. To
make this statement more precise, we quickly review some definitions and results about polyhedral sets.  
A polyhedron (or polyhedral set) in $\mathbb{R}^n$ is the intersection of a finite number of closed halfspaces. Therefore, a polyhedron $P$ can be represented as $P = \{x \in \mathbb{R}^n | Ax \leq b\}$, where $A$ is a $n \times p$ matrix and $b$ is a $p$-vector. The dimension of a polyhedron is equal to the dimension of its affine hull. A subset $F$ of a polyhedron $P$ is called a face of $P$ if there is a vector $y \in \mathbb{R}^m$ such that

$$F = \{x \in P \mid y^T x \geq y^T z \text{ for every } z \in P\}.$$ 

A face of a polyhedron is a polyhedron as well. A proper face of a polyhedron is a face whose dimension is strictly less than that of the polyhedron. A facet is an $(n-1)$-dimensional face of an $n$-dimensional polyhedron. A finite collection $\Xi = \{X_1, X_2, \ldots, X_N\}$ of polyhedral sets in $\mathbb{R}^n$ is a polyhedral subdivision of $\mathbb{R}^n$ if every polyhedron in $\Xi$ has dimension $n$, the union of all polyhedra in $\Xi$ equals $\mathbb{R}^n$, and the intersection of any two polyhedra in $\Xi$ is either empty or a common proper face of both polyhedra.

As shown in [7, Prop. 2.2.3], for a given continuous piecewise affine function $f$ there are a finite number of polyhedral sets $P_k$, $k = 1, 2, \ldots, N$, and corresponding matrices $Q_k \in \mathbb{R}^{m \times n}$ and vectors $g_k \in \mathbb{R}^m$, such that $\{P_1, P_2, \ldots, P_N\}$ is a polyhedral subdivision of $\mathbb{R}^n$ and

$$f(x) = Q_k x + g_k \quad \forall x \in P_k.$$ 

Since $f$ is continuous, we have that if $x$ is the intersection of $P_k$ and $P_\ell$ for some $k$ and $\ell$, then $Q_k x + g_k = Q_\ell x + g_\ell$.

A continuous piecewise affine system is a system of the form

$$\dot{x}(t) = f(x(t)) + Ed(t), \quad (1a)$$

$$z(t) = Hx(t), \quad (1b)$$

where $x \in \mathbb{R}^n$ is the state, $d \in \mathbb{R}^{n_d}$ is the unknown disturbance, $z \in \mathbb{R}^n$ is the output to be controlled, $E$ and $H$ are matrices of appropriate sizes, and $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a continuous piecewise affine function.

Since the right-hand side of (1a) is Lipschitz continuous in the variable $x$, for each $x_0$ and locally integrable disturbance $d$ there exists a unique absolutely continuous function $x_{x_0,d}(t)$ satisfying $x(0) = x_0$ and (1a) for almost all $t$. We denote the corresponding output by $z_{x_0,d}(t)$.

As stated above, the function $f$ admits a polyhedral subdivision. So there are polyhedral regions $X_k$, matrices $A_k$ and vectors $g_k$, for $k = 1, 2, \ldots, N$, such that $\{X_1, X_2, \ldots, X_N\}$ is a polyhedral subdivision of $\mathbb{R}^n$ and

$$f(x) = A_k x + g_k \quad \forall x \in X_k.$$ 

Hence, we can write system (1) as

$$\dot{x}(t) = A_k x(t) + g_k + Ed(t) \quad \forall x \in X_k, \quad (2a)$$

$$z(t) = Hx(t). \quad (2b)$$

We can exploit the continuity of $f(x)$ to obtain relations between the matrices $A_k$. For any two polyhedral regions $X_k$ and $X_\ell$ that share a facet $F_{k \ell} := X_k \cap X_\ell$, we can choose a vector $c_{k \ell}$ and a scalar $f_{k \ell}$ such that the affine hull of $F_{k \ell}$ is given by the hyperplane

$$\mathcal{H}_{k \ell} := \{x \in \mathbb{R}^n \mid c_{k \ell}^T x + f_{k \ell} = 0\}.$$ 

The continuity of $f$ implies that for all $x \in F_{k \ell} \subseteq \mathcal{H}_{k \ell}$ we have $A_k x + g_\ell = A_\ell x + g_k$, or equivalently $(A_\ell - A_k)x + g_k - g_\ell = 0$. Since $F_{k \ell}$ is $(n-1)$-dimensional, it follows that $\ker c_{k \ell} \subseteq \ker(A_\ell - A_k)$. Hence, there is a vector $h_{k \ell} \in \mathbb{R}^n$ such that

$$A_k - A_\ell = h_{k \ell} c_{k \ell}^T. \quad (3)$$

By combining this with the fact that $c_{k \ell}^T x + f_{k \ell} = 0$ for all $x \in F_{k \ell}$, we find that $g_k$ and $g_\ell$ satisfy

$$g_k - g_\ell = h_{k \ell} f_{k \ell}. \quad (4)$$

Notice that, since facet $F_{k \ell}$ is equal to facet $F_{k \ell}$, we can assume that $c_{k \ell} = c_{k \ell}$, $f_{k \ell} = f_{k \ell}$ and $h_{k \ell} = -h_{k \ell}$.

In some cases, it is more convenient to write system (2) in the following alternative way:

$$\dot{x}(t) = Ax(t) + Ed(t) + g(y), \quad (5a)$$

$$y(t) = Cx(t), \quad (5b)$$

$$z(t) = Hx(t), \quad (5c)$$

where $x$, $z$, $E$ and $H$ are as before, $y \in \mathbb{R}^{n_y}$ is the measured output, $A$ and $C$ are matrices of appropriate sizes, and $g : \mathbb{R}^{n_y} \rightarrow \mathbb{R}^{n_x}$ is a continuous piecewise affine function. In this representation, there is a polyhedral subdivision of the domain $\mathbb{R}^{n_y}$ of $g$: there are solid (i.e., $n_y$-dimensional) polyhedral regions $Y_k$, matrices $F_k \in \mathbb{R}^{n_y \times n_x}$ and vectors $g_k \in \mathbb{R}^{n_y}$, for $k = 1, 2, \ldots, N$, such that $\{Y_1, Y_2, \ldots, Y_N\}$ is a polyhedral subdivision of $\mathbb{R}^{n_y}$ and

$$g(y) = F_k y + g_k, \quad \text{if } y \in Y_k.$$ 

If $Y_k$ and $Y_\ell$ share a facet $F_{k \ell}$, there is a vector $\hat{c}_{k \ell} \in \mathbb{R}^{n_y}$ and scalar $\hat{f}_{k \ell}$ such that

$$F_{k \ell} \subseteq \{y \in \mathbb{R}^{n_y} \mid \hat{c}_{k \ell}^T y + \hat{f}_{k \ell} = 0\}.$$ 

Since $g$ is continuous, we can employ the same reasoning as above to see that there is a vector $\hat{h}_{k \ell}$ such that

$$F_k - F_\ell = \hat{h}_{k \ell} \hat{c}_{k \ell}^T, \quad \text{with } g_k - g_\ell = \hat{h}_{k \ell} \hat{f}_{k \ell}.$$ 

To see the equivalence between the two representations, notice that we can write system (2) in the form of system (5) by taking $C = I$, $A = A_1$, $Y_k = X_k$ and $F_k = A_k - A_1$ for $k = 1, 2, \ldots, N$. On the other hand, we can write system (5) in the form of system (2) by letting $A_k = A + F_k C$ and $X_k = C^{-1} Y_k$ for $k = 1, 2, \ldots, N$, and using the same $g_k$, $E$ and $H$. It can be shown that resulting set $\{X_1, X_2, \ldots, X_N\}$ is a polyhedral subdivision of $\mathbb{R}^n$. The corresponding facets $F_{k \ell}$ are equal to $C^{-1} F_{k \ell}$, with $c_{k \ell} = c_{k \ell} C$ and $f_{k \ell} = \hat{f}_{k \ell}$.

Combinations of linear systems and static (piecewise linear) nonlinearities, such as saturation, dead-zone and backlash, lead naturally to piecewise affine systems. A concrete example of a continuous piecewise affine system is given next.
Example 1 ([9, Example 2.2]) In high-accuracy motion control of a DC servo system, one has to deal with deadzone-type nonlinear relations between the motor torque $T$ and the current $i$ through the motor windings (see e.g. [14]). This can be modeled by the continuous piecewise affine function

$$g(y) = \begin{cases} 
k_T y - T_ - - T_t & \text{if } k_T y \leq T_ - \\
k_T y - T_+ - T_t & \text{if } T_ - \leq k_T y \leq T_+ \\
k_T y - T_+ - T_t & \text{if } T_+ \leq k_T y,
\end{cases}$$

with $k_T$ the torque constant, $T_t$ the torque applied to the rotor, and $T_-$ and $T_+$ constant values. If we assume that $T_t$ is constant, we can describe the dynamics of the current $i$ through the motor windings and the angular position $\theta$ of the rotor with the piecewise affine system

$$\begin{align*}
\frac{d}{dt} \begin{bmatrix} i \\ \theta \\ \omega \end{bmatrix} &= \begin{bmatrix} -R & 0 & -k_T \\ 0 & 0 & 1 \\ 0 & 0 & -B_T \end{bmatrix} \begin{bmatrix} i \\ \theta \\ \omega \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ g(y) \end{bmatrix}, \\
y &= \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} i \\ \theta \\ \omega \end{bmatrix},
\end{align*} \tag{6a} \tag{6b}$$

where $\omega = \frac{d}{dt} \theta$, $J$ is the moment of inertia of the rotor, and $L$, $R$, $k_T$, and $B_T$ are some constants (see [9] for details).

III. THE DISTURBANCE DECOUPLING PROBLEM

We say that a piecewise affine system, given by (1), (2) or (5), is disturbance decoupled if for all initial states $x_0 \in \mathbb{R}^{n_x}$, locally integrable disturbances $d_1$ and $d_2$, and $t \geq 0$ we have

$$z^{x_0,d_1}(t) = z^{x_0,d_2}(t).$$

In this section, we give a necessary condition for a piecewise affine system to be disturbance decoupled, as well as a sufficient condition. Both conditions are geometric in nature.

Theorem 2 If the system (2) is disturbance decoupled, then

$$\sum_{k=1}^{N} \langle A_k \mid \text{im} E \rangle \subseteq \ker H. \tag{7}$$

Proof. Let $k \in \{1, 2, \ldots, N\}$, and let $d_1(t) = d \in \mathbb{R}^{n_d}$ and $d_2 = 0$ be two distinct constant disturbances. Choose an interior point $x_0$ of $\mathcal{X}_k$, let $x_i(t)$ denote the trajectory $x^{x_0,d_i}(t)$ for $i = 1, 2$, and let $z_i(t)$ denote the corresponding output. Since $x_1$ and $x_2$ are continuous, there exists an $\varepsilon > 0$ such that $x_1(t)$ and $x_2(t)$ stay in $\mathcal{X}_k$ for $t \in [0, \varepsilon)$. Thus, the trajectories $x_1$ and $x_2$ satisfy

$$\dot{x}_i(t) = A_k x_i(t) + g_k + E d_i(t), \quad \text{for } t \in [0, \varepsilon], \quad i = 1, 2.$$

As the system is disturbance decoupled, we have that $z_1(t) = z_2(t)$ and hence $H x_1(t) = H x_2(t)$ for all $t \geq 0$. Since $d_1$ and $d_2$ are constant, we can differentiate this equation $p \geq 1$ times and obtain

$$H A_k^p x_1(t) + H A_k^{p-1} E d = H A_k^p x_2(t),$$

for all $t \in [0, \varepsilon)$. Using $t = 0$ and $x_1(0) = x_2(0)$ we get

$$H A_k^p E d = 0 \quad \forall p \geq 0.$$

Since this holds for all vectors $d \in \mathbb{R}^{n_d}$, we conclude that $H A_k^p E = 0$ for all $p \geq 0$, and hence $(A_k \mid \text{im} E) \subseteq \ker H$. By letting $k$ vary over $\{1, \ldots, N\}$, we see that (7) holds.

For the alternative representation of the system, given by (5), we have the following equivalent necessary condition for disturbance decoupling.

Corollary 3 If the system (5) is disturbance decoupled, then

$$\sum_{k=1}^{N} \langle A + F_k c \mid \text{im} E \rangle \subseteq \ker H. \tag{8}$$

In general, the subspace $\sum_{k=1}^{N} \langle A_k \mid \text{im} E \rangle$ is not necessarily invariant under $A_k$ for all $k = 1, 2, \ldots, N$. The following theorem shows that such joint invariance relations lead to a sufficient condition.

Theorem 4 The system (2) is disturbance decoupled if there is a subspace $\mathcal{V} \subseteq \ker H$ that contains $\text{im} E$ and that is invariant under $A_k$ for $k = 1, 2, \ldots, N$.

Proof. Let $r$ be the dimension of $\mathcal{V}$ and write $x^T = (e^T, w^T)$, where $e$ consists of the first $r$ entries of $x$. Note that a piecewise affine function is still piecewise affine after a basis transformation. As the property of disturbance decoupling is invariant under basis transformations as well, we can assume without loss of generality that the vectors $(e^T, 0^T)^T$ correspond to the subspace $\mathcal{V}$.

Since $\mathcal{V}$ is invariant under each $A_k$ and satisfies $\text{im} E \subseteq \mathcal{V} \subseteq \ker H$, the system matrices are of the form

$$E = \begin{bmatrix} E_1 \\ 0 \end{bmatrix}, \quad H = [0 \ H_2],$$

$$A_k = \begin{bmatrix} A_{k1}^{11} & A_{k1}^{12} \\ 0 & A_{k2}^{22} \end{bmatrix}, \quad g_k = \begin{bmatrix} g_k^1 \\ g_k^2 \end{bmatrix}, \quad k = 1, 2, \ldots, N,$$

where $A_{11}^{11} \in \mathbb{R}^{r \times r}$, $g_k^1 \in \mathbb{R}^r$, $E_1 \in \mathbb{R}^{r \times n_d}$ and $H_2 \in \mathbb{R}^{n_x \times (n_e - r)}$. Hence, we can write system (2) as

$$\begin{align*}
\dot{v} &= A_{k1}^{11} v + A_{k1}^{12} w + E_1 d + g_k^1 \ \forall (e^T, w^T)^T \in \mathcal{X}_k, \\
w &= A_{k2}^{22} w + g_k^2 \ \forall (e^T, w^T)^T \in \mathcal{X}_k, \\
z &= H_2 w.
\end{align*}$$

We see that the output $z$ depends only on $w$ and that $w$ does not directly depend on the disturbance $d$. However, the disturbance might influence the switching behavior of $x$ and in this way influence the evolution of $w$. In the rest of the proof, we will show that this is not the case.

Since $\mathcal{V}$ is invariant under all $A_k$, we have $(A_i - A_j) \mathcal{V} \subseteq \mathcal{V}$ for each $i$ and $j$. In particular, when $X_i$ and $X_j$ share a facet $F_{ij}$, we can use equation (3) to find that

$$h_{ij} c_{ij}^T \mathcal{V} \subseteq \mathcal{V}.$$

Hence, we have $h_{ij} \in \mathcal{V}$ or $\mathcal{V} \subseteq \ker c_{ij}^T$. Write $h_{ij} = (h_{ij}^{T1}, h_{ij}^{T2})$ and $c_{ij} = (c_{ij}^T, c_{ij}^T)$, where $h_{ij}^{T1}$ and $c_{ij}^T$ are in $\mathbb{R}^r$. Note that if $h_{ij} \in \mathcal{V}$, then $h_{ij}^{T1} = 0$. Consequently, using equations (3) and (4), we see that $A_{k1}^{12} = A_{k2}^{22}$ and $g_k^1 = g_k^2$. We use this observation to define clusters of modes.
We partition \( \{1,2,\ldots,N\} \) into equivalence classes as follows: \( i \) and \( j \) are in the same equivalence class if \( A_i^2 = A_j^2 \) and \( g_i^2 = g_j^2 \). Let \( I_1, I_2, \ldots, I_p \) denote the resulting equivalence classes and define clusters \( C_1, C_2, \ldots, C_p \) as
\[
C_k = \bigcup_{i \in I_k} X_i
\]
for \( \ell = 1,2,\ldots,p \). Although the union of the clusters is equal to \( \mathbb{R}^{n_x} \), \( \{C_1, C_2, \ldots, C_p\} \) is not necessarily a polyhedral subdivision of \( \mathbb{R}^{n_x} \), since a cluster is not necessarily convex. However, we see that \( w \) satisfies
\[
w = A_i^2 x + g_i^2 \quad \forall x \in C_i.
\]
Note that in the case that there is just one distinct cluster, equal to \( \mathbb{R}^{n_x} \), then \( w \) satisfies an autonomous affine system, which implies that the system is disturbance decoupled.

If there are two or more clusters, note that for any facet \( F_{ij} \) for which \( i \) and \( j \) are not in the same equivalence class, we have \( A_i^2 \neq A_j^2 \) or \( g_i^2 \neq g_j^2 \). Both inequalities imply that \( h_{ij} \neq 0 \), so \( h_{ij} \) is not an element of \( \mathcal{V} \). Consequently, the normal \( c_{ij} \) of \( F_{ij} \) satisfies \( \mathcal{V} \subseteq \text{ker} c_{ij}^T \), and hence \( c_{ij} \) must be of the form \( c_{ij}^T = [0 \; c_{ij,2}]^T \). Since there are at least two clusters, there is at least one such cluster-separating facet.

Suppose that a point \( a \) is in cluster \( C_k \), but another point \( b \) is not. Then the line segment between \( a \) and \( b \) must intersect a cluster-separating facet \( F_{ij} \) for some \( i \) and \( j \). For this facet, \( c_{ij}^T x + f_{ij} \) and \( c_{ij}^T b + f_{ij} \) have different signs. Hence, although the clusters might not be convex, for every \( x \in \mathbb{R}^{n_x} \) we can determine in which cluster \( x \) resides by only checking the values of \( c_{ij}^T x + f_{ij} \) for each cluster-separating facet \( F_{ij} \). For such facets, we have \( c_{ij}^T c_{ij} + f_{ij} = c_{ij,2} w + f_{ij} \). Consequently, the value of \( w \) completely determines which cluster \( x \) is in:
\[
w = A_i^2 x + g_i^2 \quad \text{for} \quad \begin{pmatrix} 0 \\ w \end{pmatrix} \in C_i.
\]
Thus we see that \( w \) satisfies an autonomous piecewise affine differential equation.

We are now in a position to prove that system (1) is disturbance decoupled. Let \( x_{i0} \) be any initial condition and let \( d_1 \) and \( d_2 \) be two locally integrable disturbances. Denote the two corresponding trajectories by \( x_i(t) = x_{i0},d_i(t) \), \( i = 1,2 \), and write \( x_i^T = (x_i^T, w_i^T) \). From (9) we see that \( w_i(t) = w_2(t) \) for all \( t \geq 0 \). Consequently, we have \( z_i(t) = z_2(t) \) for all \( t \geq 0 \), and hence the system is disturbance decoupled.

For the alternative representation of the system, given by (5), we have the following corollary.

**Corollary 5** The system (5) is disturbance decoupled if there is a subspace \( \mathcal{V} \subseteq \text{ker} H \) that contains \( \text{im} E \) and that is invariant under \( A + F_k C \) for \( k = 1,2,\ldots,N \).

**IV. NECESSARY AND SUFFICIENT CONDITIONS**

The sufficient conditions for system (2) to be disturbance decoupled, as given by Theorem 4, do not coincide in general with the necessary conditions provided by Theorem 2. In this section, we identify a number of particular cases for which they do coincide.

**Corollary 6** If \( \sum_{i=1}^N \langle A_i, \text{im} E \rangle \) is invariant under \( A_i \) for all \( i = 1,2,\ldots,N \), then system (2) is disturbance decoupled if and only if
\[
\sum_{i=1}^N \langle A_i, \text{im} E \rangle \subseteq \text{ker} H.
\]

**Proof.** Theorem 2 implies the necessity of the condition. For the sufficiency, let \( \mathcal{V} = \sum_{i=1}^N \langle A_i, \text{im} E \rangle \). Then \( \text{im} E \subseteq \mathcal{V} \), and by assumption we have \( \mathcal{V} \subseteq \text{ker} H \) and \( A_i \mathcal{V} \subseteq \mathcal{V} \) for all \( i = 1,2,\ldots,N \). Hence, using Theorem 4, we see that system (2) is disturbance decoupled.

To investigate when \( \sum_{i=1}^N \langle A_i, \text{im} E \rangle \) is invariant under \( A_1, A_2,\ldots,A_N \), we first look at the case that \( N = 2 \).

**Lemma 7** For two square matrices \( A_1 \) and \( A_2 \) satisfying \( A_1 - A_2 = he^T \), the subspace \( \langle A_1, \text{im} E \rangle + \langle A_2, \text{im} E \rangle \) is invariant under both \( A_1 \) and \( A_2 \). Furthermore, we have \( h \in \langle A_1, \text{im} E \rangle + \langle A_2, \text{im} E \rangle \), or \( \langle A_1, \text{im} E \rangle + \langle A_2, \text{im} E \rangle \subseteq \text{ker} c^T \).

**Proof.** Let \( \mathcal{V} = \langle A_1, \text{im} E \rangle + \langle A_2, \text{im} E \rangle \). Since \( \text{im} A_1 E \) and \( \text{im} A_2 E \) are both in \( \mathcal{V} \), we see that
\[
\text{im} he^T E \subseteq \mathcal{V}.
\]
This implies that either \( h \notin \mathcal{V} \), or \( c^T E = 0 \). Suppose that \( h \notin \mathcal{V} \), then we must have \( c^T E = 0 \), which gives us \( A_1 E = (A_1 + he^T) E = A_2 E \). Since \( \text{im} A_1 E \) and \( \text{im} A_2 E \) are both contained in \( \mathcal{V} \), we see that \( \text{im}(A_1^2 - A_2^2) E \subseteq \mathcal{V} \), so
\[
\text{im} he^T A_2 E = \text{im}(A_1 - A_2) A_2 E = \text{im}(A_1^2 - A_2^2) E \subseteq \mathcal{V}.
\]
Hence, since \( h \notin \mathcal{V} \), we have \( c^T A_2 E = 0 \), which implies
\[
A_2^2 = A_1 A_2 E = (A_2 + he^T) A_2 E = A_2^2 E.
\]
By continuing this argument, we see that \( c^T A_k E = c^T A_1 E = 0 \) and \( A_k^2 E = A_1^2 E \) for all \( k \geq 0 \). This implies that \( \mathcal{V} = \langle A_1, \text{im} E \rangle = \langle A_2, \text{im} E \rangle \) and \( \mathcal{V} \subseteq \text{ker} c^T \). Hence, we have \( h \in \mathcal{V} \) or \( \mathcal{V} \subseteq \text{ker} c^T \). This means that \( hc^T v \in \mathcal{V} \) for any \( v \in \mathcal{V} \). Since any \( v \in \mathcal{V} \) can be written as \( v = v_1 + v_2 \), with \( v_i \in \langle A_i, \text{im} E \rangle \), \( i = 1,2 \), we have \( A_1 v_1 = A_1 v_1 + A_1 v_2 = A_1 v_1 + A_2 v_2 + hc^T v_2 \in \mathcal{V} \) for all \( v \in \mathcal{V} \). Similarly, we get \( A_2 v \in \mathcal{V} \) for all \( v \in \mathcal{V} \). Hence, \( \mathcal{V} \) is invariant under both \( A_1 \) and \( A_2 \).

Next, we find a sufficient condition for \( \sum_{i=1}^N \langle A_i, \text{im} E \rangle \) to be invariant under all \( A_i \).

**Lemma 8** Consider system (2). If \( h_{kl} \in \sum_{i=1}^N \langle A_i, \text{im} E \rangle \) for all facets \( F_{kl} \), then \( \sum_{i=1}^N \langle A_i, \text{im} E \rangle \) is invariant under \( A_i \) for \( i = 1,2,\ldots,N \).

**Proof.** From Theorem 2 in [8] we know that for every \( i, j \in \{1,2,\ldots,N\} \) there is a finite sequence of indices \( k_1, k_2,\ldots,k_{r+1} \) such that \( k_1 = i, k_{r+1} = j \) and such that
\(X_k\) and \(X_{k+1}\) share a facet for \(s = 1, 2, \ldots, r\). Hence, we can write \(A_i\) as

\[
A_i = A_j + \sum_{s=1}^{r} h_{k_s,k_{s+1}} c^T_{k_s,k_{s+1}} v_s \in V,
\]

therefore, for any element \(v_j \in \langle A_j \mid \text{im } E \rangle\) we have

\[
A_i v_j = A_j v_j + \sum_{s=1}^{r} h_{k_s,k_{s+1}} c^T_{k_s,k_{s+1}} v_s \in V,
\]

since \(h_{k_s,k_{s+1}} \in V\) for \(s = 1, 2, \ldots, r\). Hence, we have \(A_j(A_i \mid \text{im } E) \subseteq V\) for every \(i\) and \(j\) and we can conclude that \(V\) is invariant under each \(A_i\) for \(i = 1, 2, \ldots, N\).}

We now investigate two special cases of systems for which the necessary conditions and sufficient conditions for disturbance decoupling coincide.

**Corollary 9** Consider system (5). If \(C(sI - A)^{-1} E\) is right-invertible as a rational matrix, then system (5) is disturbance decoupled if and only if

\[
\sum_{k=1}^{N} \langle A + F_k C \mid \text{im } E \rangle \subseteq \ker H.
\]

**Proof.** We begin by proving the following claim: if \(C(sI - A)^{-1} E\) is right-invertible, then so is \(C(sI - A - FC)^{-1} E\) for any matrix \(F \in \mathbb{R}^{n_x \times n_z}\). For this we use the well-known property

\[
(sI - B)^{-1} - (sI - A)^{-1} (sI - B)^{-1} (B - A)(sI - A)^{-1}.
\]

We take \(B = A + FC\) and multiply both sides with \(C\) from the left and with \(E\) from the right. Rearranging the terms then gives us

\[
C(sI - A - FC)^{-1} E
= \left( I + C(sI - A - FC)^{-1} F \right) C(sI - A)^{-1} E.
\]

Since \(I + C(sI - A - FC)^{-1} F\) and \(C(sI - A)^{-1} E\) are both right-invertible as a rational matrices, the claim follows.

Let \(V = \sum_{k=1}^{N} \langle A + F_k C \mid \text{im } E \rangle\). For any facet \(F_{k\ell}\), we have \(c^T_{k\ell} C(sI - A - F_m C)^{-1} E \neq 0\), for \(m = k, \ell\), since \(c^T_{k\ell} \neq 0\). Equivalently, \(c^T_{k\ell} (A + F_m C \mid \text{im } E) \neq \{0\}\) for \(m = k, \ell\). Hence, by Lemma 7, we see that \(h_{k\ell} \in V\) for all facets \(F_{k\ell}\). Then, by Lemma 8, \(V\) is invariant under all \(A + F_k C\). From Corollaries 5 and 6, we see that system (5) is disturbance decoupled if and only if \(V \subseteq \ker H\).}

**Corollary 10** Consider system (2). If all normals \(c_{ij}\) to facets \(F_{ij}\) are parallel, then system (2) is disturbance decoupled if and only if

\[
\sum_{i=1}^{N} \langle A_i \mid \text{im } E \rangle \subseteq \ker H.
\]

**Proof.** Let \(V = \sum_{i=1}^{N} \langle A_i \mid \text{im } E \rangle\). If all normals \(c^T_{ij}\) to facets \(F_{ij}\) are parallel, then all the facets are parallel. This means that the state space is sliced up into parallel regions, each of which shares a facet with at most two other regions.

Suppose there is a facet \(F_{ij}\) for which \(c^T_{ij}(A_i \mid \text{im } E) = \{0\}\), then \(c^T_{ij} A_p^T E = 0\) for all \(p \geq 0\). This implies that we have \(A^T E = A_p^T E\) for all \(p \geq 0\), which we prove by mathematical induction. Clearly it holds for \(p = 0\). Suppose that it holds for some \(p\), then

\[
A^{p+1} E = A_j A^p E = (A_i - h_{ij} c^T_{ij}) A^p E = A^p E + E D,
\]

which proves the claim. Consequently, we have that \(A_i \mid \text{im } E = \langle A_j \mid \text{im } E\rangle\), so \(c^T_{ij}(A_i \mid \text{im } E) = \{0\}\). Moreover, if \(X_j\) shares a facet \(F_{ij}\) with some other region \(X_k\) as well, then we see, using that \(c_{jk}\) is a multiple of \(c_{ij}\), that \(c^T_{jk} A_j \mid \text{im } E = \{0\}\). By the same reasoning as above, we get that \(A_k \mid \text{im } E = \langle A_j \mid \text{im } E\rangle\). By continuing this argument from region to region, we find that for all facets \(F_{ij}\) we have \(c^T_{ij}(A_i \mid \text{im } E) = \{0\}\) and \(A_j \mid \text{im } E = \langle A_i \mid \text{im } E\rangle\). Hence, we conclude that \(V = \langle A_i \mid \text{im } E\rangle\) for any \(i \in \{1, 2, \ldots, N\}\). As a consequence, \(V\) is invariant under \(A_i\) for all \(i = 1, 2, \ldots, N\).

Hence, in both cases \(V\) is invariant under all \(A_i\). By Corollary 6, system (5) is disturbance decoupled if and only if \(V \subseteq \ker H\).}

**Example 11** We consider the system as given in Example 1 and add a disturbance \(d\):

\[
\frac{d}{dt} \begin{bmatrix} i \\ \theta \\ \omega \end{bmatrix} = \begin{bmatrix} -\frac{B_i}{m} & 0 & -\frac{B_f}{m} \\ 0 & 0 & 1 \\ 0 & 0 & -\frac{B_i}{J} \end{bmatrix} \begin{bmatrix} i \\ \theta \\ \omega \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ g(y) \end{bmatrix} + Ed
\]

\[
y = z = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} i \\ \theta \\ \omega \end{bmatrix}^T,
\]

with \(E \in \mathbb{R}^{n_x \times 1}\). To illustrate the theory developed in this section, we discuss whether this system is disturbance decoupled for three choices for \(E\): \(E_1\), \(E_2\) and \(E_3\), where \(E_i\) is the \(i\)th column of the \(3 \times 3\) identity matrix. The system satisfies the conditions of Corollary 10. Hence, we only have to check if \(\sum_{k=1}^{N} \langle A_k \mid \text{im } E \rangle \subseteq \ker H\), where \(A_k = A + F_k C\). For \(E_2\), we have \(\langle A_i \mid \text{im } E \rangle = \text{im } E \subseteq \ker H\) for \(i = 1, 2, 3\). Therefore, \(\sum_{k=1}^{N} \langle A_k \mid \text{im } E \rangle = \ker H\), implying that the system is disturbance decoupled. For \(E_1\), we see that \(\text{im } E_1 \not\subseteq \ker H\), hence \(\sum_{k=1}^{N} \langle A_k \mid \text{im } E \rangle \not\subseteq \ker H\). For \(E_3\), we have \(\text{im } E \subseteq \ker H\), but \(\sum_{k=1}^{N} \langle A_k \mid \text{im } E \rangle = \mathbb{R}^{n_x} \not\subseteq \ker H\). Consequently, the system is not disturbance decoupled for \(E = E_1\) or \(E = E_3\).

**V. STATE FEEDBACK**

In this section, we discuss the problem of finding a state feedback law that renders a given piecewise affine system disturbance decoupled. We consider the continuous piecewise affine system

\[
\dot{x}(t) = f(x(t)) + Ed(t) + Bu(t) \tag{11a}
\]

\[
z(t) = Hx(t), \tag{11b}
\]
with \( x, d, z, E \) and \( H \) as before, \( u \in \mathbb{R}^{nu} \) the input, \( B \in \mathbb{R}^{nx \times nu} \), and \( f : \mathbb{R}^{nx} \to \mathbb{R}^{nx} \) a piecewise affine function. Like before, \( f \) admits a polyhedral subdivision \( \{X_1, X_2, \ldots, X_N\} \) of \( \mathbb{R}^{nx} \). For each region \( X_k \), there are matrices \( A_k \) and vectors \( g_k \) such that \( f(x) = A_k x + g_k \) for all \( x \in X_k \). Using this, we can write system (11) as
\[
\dot{x}(t) = A_k x(t) + g_k + Ed(t) + Bu(t) \quad \forall x \in X_k \quad (12a)
\]
\[
z(t) = H x(t). \quad (12b)
\]
In the rest of this section we will investigate conditions for the existence of a mode-independent state feedback law that renders system (12) disturbance decoupled.

We consider a mode-independent feedback law \( u = K x \), for some matrix \( K \in \mathbb{R}^{nx \times nu} \). Applying such a feedback law to system (12) results in the following closed-loop system
\[
\dot{x}(t) = (A_k + BK)x(t) + g_k + Ed(t) \quad \forall x \in X_k \quad (13a)
\]
\[
z(t) = H x(t). \quad (13b)
\]
In view of Theorem 4 we see that system (13) is disturbance decoupled if there is a subspace \( V \) that satisfies
\[
(A_k + BK)V \subseteq V \quad \forall i = 1, 2, \ldots, N \quad (14)
\]
\[
im E \subseteq \ker H. \quad (15)
\]
To investigate whether such a subspace exists, we define the following set of subspaces:
\[
V(H, \{A_k\}_{k=1}^N, B) = \{V \subseteq \ker H \mid \exists K \in \mathbb{R}^{nu \times nx} \quad s.t. \quad (A_k + BK)V \subseteq V, \forall k = 1, 2, \ldots, N\}.
\]
It is easy to see that a subspace \( V \) is an element of \( V(H, \{A_k\}_{k=1}^N, B) \) if and only if \( A_1 V \subseteq V + \ker B \) and \( (A_i - A_j)V \subseteq V \) for all \( i, j \in \{1, 2, \ldots, N\} \). For any two subspaces \( V_1, V_2 \in V \) we have that \( V_1 + V_2 \subseteq \ker H \), \( A_1 (V_1 + V_2) \subseteq V + \ker B \) and
\[
(A_i - A_j)(V_1 + V_2) \subseteq (A_i - A_j)V_1 + (A_i - A_j)V_2 \subseteq V_1 + V_2
\]
for all \( i, j \in \{1, 2, \ldots, N\} \). Therefore, we see that the set \( V(H, \{A_k\}_{k=1}^N, B) \) is closed under subspace addition. Thus we can define \( V^*(H, \{A_k\}_{k=1}^N, B) \) to be the largest element in \( V(H, \{A_k\}_{k=1}^N, B) \) that is contained in \( \ker H \). We observe that \( im E \subseteq V^*(H, \{A_k\}_{k=1}^N, B) \) if and only if there is a subspace \( V \) satisfying (14)-(15).

Hence, we arrive at the following theorem.

**Theorem 12** There exists a feedback law \( u = K x \) that renders the system (13) disturbance decoupled if
\[
im E \subseteq V^*(H, \{A_k\}_{k=1}^N, B).
\]

**Remark 13** To compute \( V^*(H, \{A_k\}_{k=1}^N, B) \), we refer to [13, Algorithm 5.3].

**Example 14** We extend on Example 1 and 2, by adding state feedback in the form of applying a voltage \( v \) to the motor.

This results in the system
\[
\frac{d}{dt} \begin{bmatrix} i \\ \theta \\ \omega \end{bmatrix} = \begin{bmatrix} -\frac{R}{L} & 0 & -\frac{k}{L} \\ 0 & 0 & 0 \\ 0 & 0 & -\frac{B}{J} \end{bmatrix} \begin{bmatrix} i \\ \theta \\ \omega \end{bmatrix} + Ed + Bu
\]
\[
y = z = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} i \\ \theta \\ \omega \end{bmatrix}^T,
\]
where for we choose \( B = [1 \ 0 \ 0]^T \) and \( E = [0 \ 0 \ 1]^T \).

Then for \( K = [0 \ 0 \ \frac{R}{L}] \), the subspace \( \ker H \) satisfies \( (A_1 + BK) \ker H \subseteq \ker H \) for \( i = 1, 2, 3 \), hence \( \ker H \subseteq V^*(H, \{A_1, A_2, A_3\}, B) \). On the other hand, \( V^*(H, \{A_1, A_2, A_3\}, B) \) is contained in \( \ker H \). Consequently, we have \( V^*(H, \{A_1, A_2, A_3\}, B) = \ker H \). Since im \( E \subseteq \ker H \), we conclude that the feedback \( u = K x \) renders the system disturbance decoupled.

**VI. CONCLUSIONS**

In this paper, we established necessary conditions as well as sufficient conditions for a piecewise affine system to be disturbance decoupled. These conditions do not coincide in general. However, we identified a number of particular cases for which they do coincide. Furthermore, we provided conditions for the existence of a mode-independent static feedback controller that renders a given piecewise affine system disturbance decoupled. All presented conditions are geometric in nature and easily verifiable.

Further research possibilities include investigating the gap between the necessary conditions and sufficient conditions as well as studying mode-dependent state feedback for disturbance decoupling.

**REFERENCES**


