Equivalence of Decentralized Stochastic Dynamic Decision Systems via Girsanov’s Measure Transformation

Charalambos D. Charalambous and N.U. Ahmed

Abstract—In this paper we present two methods for decentralized optimization of continuous and discrete-time stochastic dynamic decision systems, with multiple decision makers having nonclassical information structures. For both methods we apply Girsanov’s change of measure to transform such problems into equivalent optimization problems, under a reference probability measure, with corresponding observations and information structures available for decisions, which are not affected by any of the decisions.

The first method, when applied to continuous-time Itô Stochastic Differential Equations (SDEs), involves function space integration with respect to the Wiener measure. Its corresponding discrete-time analog generalizes Witsenhausen’s “Common Denominator Condition” and “Change of Variables”, and relates them to the Radon-Nikodym Derivative and Bayes’ theorem associated with Girsanov’s theorem.

The second method is based on stochastic Pontryagin’s maximum principle, and applies to continuous and discrete-time stochastic dynamic systems. The decentralized optimality conditions are given by a “Hamiltonian System” consisting of forward and backward SDEs, and conditional variational Hamiltonians, conditioned on the information structure of each of the decision makers.

I. INTRODUCTION

Over the last 50 years several methods are proposed to address optimality of decentralized decisions, in networked control and real-time communication systems. These methods often employ Marschak’s and Radner’s [2], [3] static team theory formalism [4]–[9]. An notable method is presented by Witsenhausen [1] “Equivalent Stochastic Control Problems”. His main observation is that for a certain class of problems, discrete-time stochastic dynamic decision problems, with nonclassical information structures and finite action spaces (including some continuous alphabet spaces), can be transformed to equivalent static team problems. This equivalence is described in terms of the so-called “Common Denominator Condition” and “Change of Variables”. Witsenhausen’s [Section 2.1, [1]] analysis is carried under the assumption that the state process is a Random Variable. Moreover, no expression is given for the common denominator condition and change of variables, which facilitate the equivalence between the two problems. In this paper we (1) apply Girsanov’s theorem [10] to transform decentralized discrete and continuous-time stochastic dynamic decision problems to equivalent decision problems under a reference probability measure, with corresponding observations and information structures generated by them which are not affected by any of the team decisions;

(2) identify, via Girsanov’s theorem and the corresponding Radon-Nikodym derivative, the Common Denominator Condition and Change of Variables, which facilitate the equivalence of the two problems, under the transformed probability measure;

(3) apply the method to the Witsenhausen counterexample [6] to derive the optimal decision strategies in closed form;

(4) propose an alternative method based on stochastic maximum principle, and give necessary conditions of optimality, in terms of a Hamiltonian system, conditional Hamiltonian, and Backward SDEs, to characterize the optimal decentralized decisions.

Decentralized Stochastic System: Continuous-Time.

The continuous-time model consists of the following elements.

1) Observations \( \{y^m(t) : t \in [0, T]\} \) generated at \( M \) observation posts \( m=1, \ldots, M \), and control inputs \( \{u^k(t) : t \in [0, T]\} \) applied at \( K \) control stations \( k=1, \ldots, K \), which affect the state and observations \( \{x(t) : t \in [0, T]\} \), \( \{y^m(t) : t \in [0, T]\} \) of the system;

2) State dynamics and observations equations described by Itô SDEs both driven by the controls

\[
\begin{align*}
\mathrm{d}x(t) &= f(t, x(t), u^1(t), \ldots, u^K(t)) \mathrm{d}t + \sigma(t, x(t), u^1(t), \ldots, u^K(t)) \mathrm{d}W(t), \quad x(0) = x_0, \\
\mathrm{d}y^m(t) &= h^m(t, x(t), u^1(t), \ldots, u^K(t)) \mathrm{d}t + D^m \frac{1}{2} (t) dB^m(t), \quad y^m(0) = 0, m = 1, \ldots, M,
\end{align*}
\]

where \( \{x(t) : t \in [0, T]\} \) is the \( \mathbb{R}^n \)-valued state process, \( \{W(t) : t \in [0, T]\} \) is an \( \mathbb{R}^d \)-valued Brownian motion state noise process, and \( \{B^m(t) : t \in [0, T]\} \) is an \( \mathbb{R}^m \)-valued Brownian motion observation noise process, \( m = 1, \ldots, M \);

3) Nonclassical information structures assigned to each control station \( k \in \{1, \ldots, K\} \) at each time \( t \in [0, T] \) of the form \( \{\gamma_{\tau}(\tau, \mu) : (\tau, \mu) \in \mathcal{Y}_{\tau,k}\} \), where \( \mathcal{Y}_{\tau,k} \subseteq \gamma_{\tau} \triangleq \{(s, m) \in [0, \tau] \times \{1, \ldots, M\}\};

4) Control laws applied at \( K \) control stations generating

The work of C.D. Charalambous was financially supported by a medium size University of Cyprus grant entitled DIMITRIS and by QNRF, a member of Qatar Foundation, under the project NPRP 6-784-2-329.

C. D. Charalambous is with the School of Electrical Engineering, University of Cyprus, Nicosia, Cyprus, charal@ucy.ac.cy.

N.U. Ahmed is with the School of Engineering and Computer Science, and Department of Mathematics, University of Ottawa, Ontario, Canada. ahmed@site.uottawa.ca.
actions via measurable maps
\[ u^k(t) = \gamma^k \left( t, \{ y^\mu(\tau) : (\tau, \mu) \in \mathcal{Y}_{t,k} \} \right) \]
≡\( \gamma^k \left( t, \Pi_{y^k_{t,k}}(Y) \right) \), \( t \in [0,T], k = 1, \ldots, K \),
where for each \((t, k) \in [0, T] \times \{1, \ldots, K\} \), \( \Pi_{y^k_{t,k}}(Y) \)
denotes the projection operator of all observations \( Y = \{ y^i(t), \ldots, y^M(t) : t \in [0, T] \} \) to any subset \( \{ y^\mu(\tau) : \tau \in [0, t], \mu \in \kappa(t) \}, \kappa(t) \subseteq \{1, \ldots, M\} \).

5) Admissible control laws at the control stations are chosen from some admissible set \( \gamma(\cdot) = \gamma_k(\cdot) : k = 1, \ldots, K \) \( \in \mathcal{U}(K)_{[0,T]} \) to optimize the common pay-off (called team optimality)
\[ \inf \left\{ J(\gamma) : \gamma \in \mathcal{U}(K)_{[0,T]} \right\}, \]
\[ J(\gamma) = \mathbb{E} \left\{ \int_0^T \ell(t, x(t), x(t), \Pi_{y^k_{t,k}}(Y)) dt + \varphi(x(T)) \right\}. \]

\( f, \sigma, h^m, D^{m, -\frac{1}{2}} \) can be nonanticipative on \( Y \) (predictable).
We also discuss the discrete-time analog of 1)-(5).

II. EQUIVALENT DECENTRALIZED STOCHASTIC DECISION SYSTEMS

In this section we venture into Girsanov’s theorem to transform the stochastic dynamic decision problem 1)-(5) to an equivalent problem, under a reference probability measure with corresponding observations and information structures, which are independent of any of the team decisions.

\( (\Omega, \mathcal{F}, \mathbb{P}) \): complete filtered probability space satisfying the usual conditions [10].
\( C([0,T], \mathbb{R}^n) \): space of continuous real-valued \( n \)-dimensional functions defined on the interval \([0,T] \).
\( L^2_{\mathcal{F}_{0,t}}([0,T], \mathbb{R}^n) \): space of \( \mathcal{F}_{0,t} \)-adapted \( n \)-dimensional random processes \( \{ z(t) : t \in [0,T] \} \) such that \( \mathbb{E} \int_0^T |z(t)|^2_{\mathbb{R}^n} dt < \infty \).
\( L^2_{\mathcal{F}_{0,t}}([0,T], \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n)) \): space of \( \mathcal{F}_{0,t} \)-adapted \( n \times m \) matrix valued random processes \( \{ \Sigma(t) : t \in [0,T] \} \) such that \( \mathbb{E} \int_0^T |\Sigma(t)|^2_{\mathcal{L}(\mathbb{R}^m, \mathbb{R}^n)} dt \leq \infty \).
\( B^2_{\mathcal{F}_{0,t}}([0,T], \mathbb{R}^n) \): space of \( \mathcal{F}_{0,t} \)-adapted \( n \)-dimensional second order random processes \( \{ z(t) : t \in [0,T] \} \) such that \( \| z \|^2 \triangleq \sup_{t \in [0,T]} \mathbb{E} [z(t)]^2_{\mathbb{R}^n} \).

We introduce the following regularity conditions.

Assumption II.1 (Main Assumptions)
\( A^k \subseteq \mathbb{R}^{d_k}, \forall k \in \mathbb{Z}_K \) are nonempty, and the maps
\[ f : [0,T] \times \mathbb{R}^n \times A(K) \rightarrow \mathbb{R}^n, \]
\[ \sigma : [0,T] \times \mathbb{R}^n \times A(K) \rightarrow \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n), \quad A(K) \triangleq \times_{k=1}^K A_k, \]
\[ h^m : [0,T] \times \mathbb{R}^n \times A(K) \rightarrow \mathbb{R}^{m}, \forall m \in \mathbb{Z}_M, \]
\[ D^{m,\frac{1}{2}} : [0,T] \rightarrow \mathcal{L}(\mathbb{R}^{km}, \mathbb{R}^{km}), \forall m \in \mathbb{Z}_M, \]
\[ \varphi : \mathbb{R}^n \rightarrow \mathbb{R}, \quad \ell : [0,T] \times \mathbb{R}^n \times A(K) \rightarrow \mathbb{R}, \]
satisfy the following conditions.

(C1) \( \{ f, \sigma, \{ h^m : m = 1, \ldots, M \} \} \) are continuous with respect to \( (t, x, u) \in [0,T] \times \mathbb{R}^{m} \times A(K) \) and once continuously differentiable in \((x,u)\);

(C2) The first derivatives \( \{ f_x, \sigma_x, f_u, \sigma_u, \{ h^m_x, h^m_u : m = 1, \ldots, M \} \} \) are bounded uniformly on \([0,T] \times \mathbb{R}^{m} \times A(K) \);

(C3) There exists a \( C > 0 \) such that \( h^m(t, x, u)||_{\mathbb{R}^{km}} \leq C \), \( \forall (t, x, u) \in [0,T] \times \mathbb{R}^{m} \times A(K) \), for \( m = 1, \ldots, M \).

(C4) \( D^{m,\frac{1}{2}} \triangleq D^{k,\frac{1}{2}}(D^{k,\frac{1}{2}})^\ast, D^{m,\frac{1}{2}} \) is measurable, uniformly bounded, the inverse \( D^{m,\frac{1}{2}} \) exists and it is uniformly bounded, for \( m = 1, 2, \ldots, M \).

(C5) The map \( \ell \) is continuous with respect to \((t, x, u) \in [0,T] \times \mathbb{R}^{m} \times A(K) \), once continuously differentiable with respect to \((x,u)\), the map \( \varphi \) is continuous with respect to \( x \in \mathbb{R}^{n} \), once continuously differentiable with respect to \( x \), and there exists a \( C > 0 \) such that
\[ (1 + |x|_{\mathbb{R}^n} + |u|_{\mathbb{R}^d})^{-1} |\ell(t, x, u)|_{\mathbb{R}} \]
\[ + (1 + |x|_{\mathbb{R}^n} + |u|_{\mathbb{R}^d})^{-1} |\ell_x(t, x, u)|_{\mathbb{R}^n} \]
\[ + (1 + |x|_{\mathbb{R}^n} + |u|_{\mathbb{R}^d})^{-1} |\ell_u(t, x, u)|_{\mathbb{R}^d} \]
\[ + (1 + |x|_{\mathbb{R}^n})^{-1} |\varphi(x)|_{\mathbb{R}^n} + (1 + |x|_{\mathbb{R}^n})^{-1} |\varphi_x(x)|_{\mathbb{R}^n} \leq C; \]

Equivalent Decentralized Stochastic System: Continuous-Time-(\( \Omega, \mathcal{F}, \mathbb{P}_0 \) : t \in [0,T], \( \mathbb{P}_n \)).
We start with a decentralized stochastic systems, defined on a reference probability space \( (\Omega, \mathcal{F}, \mathbb{P}) \), with corresponding observations and information structures which are not affected by team decisions, and then show its equivalence to the stochastic systems of Section I. 1)-5).

(WP1) \( x(0) = x_0 \): an \( \mathbb{R}^n \)-valued Random Variable with distribution \( \Pi_0(dx) \);
(WP2) \( \{ W(t) : t \in [0,T] \} \): an \( \mathbb{R}^m \)-valued standard Brownian motion, independent of \( x(0) \);
(WP3) \( \{ B^m(t) : t \in [0,T] \} \): an \( \mathbb{R}^{km} \)-valued, \( m = 1, \ldots, M \), mutually independent Standard Brownian motions, independent of \( \{ W(t) : t \in [0,T] \} \), and \( M \) observation posts with observations
\[ y^m(t) = \mathbb{E} \int_0^t D^{m,\frac{1}{2}}(s)dB^m(s), \forall m \in \mathbb{Z}_M. \]

Let \( \{ F_{0,t}^W : t \in [0,T] \}, \{ G_{0,t}^y : t \in [0,T] \} \) denote the complete (with respect to \( \mathbb{P} \) filtrations generated by \( \{ W(t) : t \in [0,T] \}, \{ y^m(t) : t \in [0,T] \} \), respectively, for \( i = 1, \ldots, M \), and let \( F^{x(0)}_0 \triangleq \sigma(x(0)) \). Define
\[ G^y_{0,t} \triangleq \bigvee_{m=1}^M G^y_{0,t} \triangleq F^{x(0)} \lor F_{0,t}^W \lor G^y_{0,t} \lor F_{0,t}^W. \]

Information Structures. These are defined in Section I, 3), 4), following [7]. The control applied by the \( k \)th station at time \( t \) has argument \( \{ y^\mu(\tau) : (\tau, \mu) \in \mathcal{Y}_{k,t} \} \), and hence, it is a Borel measurable map \( \gamma_k(\cdot) \), defined by
\[ u^k(t) = \gamma_k^k \left( \{ y^\mu(\tau) : (\tau, \mu) \in \mathcal{Y}_{k,t} \} \right), t \in [0,T]. \]
For each \((t,k) \in U_T \triangleq \{(\tau,k) \in [0,T] \times \{1,\ldots,K\}\},
\)
given the data basis \(Y_{t,k}\), let
\(G_{t,k} = \sigma\{y_{\mu}(\tau) \mid \mu \in \mathcal{Y}_{t,k}\},\)
for \(k \in \mathbb{Z}_K, t \in [0,T]\).

The control process applied at the kth control station at time \(t \in [0,T]\) is an \(G_{t,k}^{\mathcal{T}}\) measurable function.

**Perfect Recall.** Control station \(k \in \{1,\ldots,K\}\) is said to have “Perfect Recall” at \(t \in [0,T]\), if \(Y_{t,k} \subseteq Y_{\tau,k}, \forall \tau \geq t, \tau \in [0,T]\).

Perfect recall means that a station that at some time has available certain information will have available the same information at any subsequent times.

**Classical.** An information structure is called “Classical” if the following two conditions hold: i) all stations receive the same information (i.e. the information structures are independent of \(k \in \{1,\ldots,K\}\)), ii) all stations have perfect recall for all \(t \in [0,T]\).

**Nonclassical.** An information structure is called “Nonclassical” if any of the two conditions i), ii) fails.

If the control stations have classical information structures, then each \(\sigma\)–algebra generated by the corresponding information structures at the control stations over successive times are nested and all information structures are identical.

**Definition II.2 (Admissible Strategies)**

The admissible decisions applied at the kth control station are square integrable processes, and for each \(t \in [0,T]\) are \(G_{t,k}^{\mathcal{T}}\) measurable functions taking values in \(\mathbb{K}_k\); defined by
\[ U_k^{0}[0,T] \triangleq \left\{ u_k \in L^2_{G_{t,k}}([0,T], \mathbb{R}^{d_k}) : u_k(t) \in \mathbb{K}_k \subseteq \mathbb{R}^{d_k}, \right. \]
\[ \left. a.e.t \in [0,T], \mathbb{P} - a.s. \right\}, \forall k \in \mathbb{Z}_K. \]

A K tuple of strategies is defined by \(u \triangleq (u^1, u^2, \ldots, u^K) \in U^{|K|}[0,T] \triangleq \times_{k=1}^N U_k^{0}[0,T]. \)

Since \(\{y_m(\cdot) : m = 1,\ldots,M\}\) are Brownian motions, for each \((t,k) \in U_T\) then \(G_{t,k}^{\mathcal{T}}\) are fixed and independent of the decisions, and hence the \(G_{t,k}^{\mathcal{T}}\) measurable decisions \(u_k(t)\) are independent of the other decisions.

On \((\Omega, \mathbb{F}, \mathbb{P})\), for each \(u \in U^{|N|}[0,T]\), we denote by \(x(t) \equiv x^u(t) : t \in [0,T]\) the solution of the SDE (i.e. that of (1))
\[ dx(t) = f(t, x(t), u(t))dt + \sigma(t, x(t), u(t))dW(t). \]

By Assumptions II.1, (C1), (C2), for any \(x(0)\) with finite second moment, \(\{x(t) : t \in [0,T]\}\) is pathwise unique \(\{\mathbb{F}_{0,t} : t \in [0,T]\}\) adapted \(C[0,T], \mathbb{R}^n\)–martingale, and satisfies \(x(\cdot) \in B_{\mathbb{F}_{0,T}}^\mathcal{T}([0,T], L^2(\Omega, \mathbb{R}^n))\).

For any \(u \in U^{|N|}[0,T]\) we introduce the exponential functions
\[ \Lambda_{m,u}(t) \triangleq \exp\left\{ \int_0^t h_{m,s}(x(s), u(s))D^{m-1}(s)dy_m(s) \right. \]
\[ \left. - \frac{1}{2} \int_0^t h_{m,s}(x(s), u(s))D^{m-1}(s)h_{m}(x(s), u(s))ds \right\}, \]
\[ \Lambda^u(t) \triangleq \prod_{m=1}^M \Lambda_{m,u}(t), \quad \Lambda^u(0) = 1, \quad t \in [0,T]. \]

Under the additional Assumptions II.1, (C3), (C4) the process \(\Lambda^u(t) : t \in [0,T]\) is a martingale and the unique \(\{\mathbb{F}_{0,t} : t \in [0,T]\}\) adapted continuous solution of the SDE
\[ d\Lambda^u(t) = \Lambda^u(t) \sum_{m=1}^M h_{m,s}(x(t), u(t))D^{m-1}(t)dy_m(t). \]

The equivalent decentralized stochastic systems is defined under the reference probability space \((\Omega, \mathbb{F}, \mathbb{P})\), by (7), (9), (6), (10), and pay-off
\[ J(u) \triangleq \mathbb{E}\left\{ \int_0^T \Lambda^u(t)f(t, x(t), u(t))dt + \Lambda^u(t)\varphi(x(T)) \right\}. \]

**Initial Decentralized Stochastic System: Continuous-Time** \((\Omega, \mathbb{F}, \{\mathbb{F}_{0,t} : t \in [0,T]\}, \mathbb{P}^u)\). Next, we show (7), (9), (6), (10) is indeed equivalent to the initial decentralized stochastic system described in Section I, 1)-5).

By (C3), for any \(u \in U^{|N|}[0,T]\), \(\Lambda^u(\cdot)\) defined by (8) is an \(\{\mathbb{F}_{0,t} : t \in [0,T]\}\) martingale, hence it has constant expectation, \(\int_0^T \Lambda^u(t, \omega)d\mathbb{P}(\omega) = 1, \forall t \in [0,T]\).

Therefore, we can introduce a probability measure \(\mathbb{P}^u\) on \((\Omega, \{\mathbb{F}_{0,t} : t \in [0,T]\})\) by setting \(d\mathbb{P}^u \triangleq \Lambda^u(T)\). By Girsanov’s theorem, under the probability space \((\Omega, \mathbb{F}, \mathbb{P}^u)\), the process \(B^{m,u}(t) : t \in [0,T]\) defined by
\[ B^{m,u}(t) \triangleq B^m(t) \]
\[ - \int_0^t D^{m-\frac{1}{2}}(s)h_m(s, x(s), u(s))ds, \forall m \in \mathbb{Z}_M, \]

is a standard Brownian motion, and finally, by (6) and (11) the observations are defined by
\[ dy_m(t) = h_m(t, x(t), u(t))dt + D^{m-\frac{1}{2}}(t)dB^{m,u}(t), \]

and the state process \(\{x(t) : t \in [0,T]\}\) is defined by (7) (its distribution is unchanged). Moreover, we have the following.

The decentralized stochastic system is defined under the original probability space \((\Omega, \mathbb{F}, \mathbb{P})\) by (7), (12), and pay-off
\[ J(u) = \mathbb{E}^u \left\{ \int_0^T f(t, x(t), u(t))dt + \varphi(x(T)) \right\}. \]

**Conclusion II.3** On \((\Omega, \mathbb{F}, \{\mathbb{F}_{0,t} : t \in [0,T]\}, \mathbb{P}^u)\) the state process, observations process and pay-off are (7), (12), (13), while under the reference probability space \((\Omega, \mathbb{F}, \{\mathbb{F}_{0,t} : t \in [0,T]\}, \mathbb{P})\) they are (7), (9), (6), (10), i.e., \(\{y_m(\cdot) : t \in [0,T]\}\), \(m = 1,\ldots, M\) are independent Brownian motions, and hence any information structure generated by them is independent of any of the team decisions.

**A. Team Optimality via Function Space Integration**

Now we venture into an applications of function space integration over Wiener measures, for team optimality.

**Assumption II.4** Let \(K = M \equiv N\) and replace \(\sigma\) in (7) by
\[ ^1G\] can be allowed to depend on \(x\).
Then the transformed pay-off (16) is given by
\[
J(u) \triangleq \mathbb{E} \left\{ \int_{0}^{T} \left( \int_{0}^{t} \bar{f}(t, \xi, \{\bar{W}(s), y(s), u(s) : 0 \leq s \leq t\})dt + \bar{g}(T, \xi, \{\bar{W}(t), y(t), u(t) : 0 \leq t \leq T\}) \right) \right\}
\]
\[
\times \Pi_{0}(d\xi) \times \mathcal{W}_{g}(d\bar{W}) \times \mathcal{W}_{y}(dy)
\]
(17)
\[
= \int_{C([0,T],\mathbb{R}^{k})}^{T} \int_{0}^{T} V(t, \{y(s), u(s) : 0 \leq s \leq t\})dt + N(T, \{y(t), u(t) : 0 \leq t \leq T\}) \right\} \mathcal{W}_{y}(dy).
\]
(18)
The equivalent pay-off (18) is a function space integral with respect to a Wiener measure [11], [12].

Conclusion II.6 Expression (18) is precisely the continuous-time generalization of Wisenhausen's main theorem [Theorem 6.1, [11]], i.e., compare (18) and (equation 6.4), [11]. In fact, $\mathcal{Y}^u(\cdot)A^u(\cdot)$ is the common denominator, in Wisenhausen's [11] terminology.

III. EQUIVALENT DECENTRALIZED STOCHASTIC DECISION SYSTEMS: DISCRETE-TIME

In this Section we apply the same methodology to discrete-time models operating over the time period $\{0, \ldots, T\}$. Let $\mathbb{N}_T^d \triangleq \{0, 1, 2, \ldots, T\}$, $\mathbb{N}_T^{d^2} \triangleq \{1, 2, \ldots, T\}$, $x_0^d \triangleq (x(0), x(1), \ldots, x(t))$, $t \in \mathbb{N}_T^d$, $u \triangleq (u^1, u^2, \ldots, u^N)$, $K = M \equiv N$, and $G(t) \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$, $t \in \mathbb{N}_T^d$, $D^2(t) \in \mathcal{L}(\mathbb{R}^k, \mathbb{R}^k)$, $t \in \mathbb{N}_T^{d^2}$ invertible matrices.

Equivalent Decentralized Stochastic System: Discrete-Time ($\Omega, \mathcal{F}, \{\mathcal{F}_{0,t} : t \in \mathbb{N}_T^d\}$, $\mathbb{P}$) We start with
\[
\{\{x(t), y^m(t) : t \in \mathbb{N}_T^d\} \text{ sequences of independent RVs x(0) has distribution } \Pi_{0}(dx),\]
\[
\{x(t) \sim \zeta(\cdot) \triangleq \text{Gaussian}(0, I_{n \times n}) : t \in \mathbb{N}_T^d\},\]
\[
\{y^m(t) \sim \lambda^m(\cdot) \triangleq \text{Gaussian}(0, I_{km \times km}) : t \in \mathbb{N}_T^d, m \in \mathbb{Z}^N\}.
\]
Let $\{\mathcal{F}_{0,t} : t \in \mathbb{N}_T^{d^2}\}$ denote the filtration generated by the $\sigma-$algebra $\sigma\{x(t), y^1(t), \ldots, y^N(t) : t \leq t\}, t \in \mathbb{N}_T^d$, and $\{G^m_{0,t} : t \in \mathbb{N}_T^d\}$ that of $\sigma-$algebra $\sigma\{y^m(t) : t \leq t\}, t \in \mathbb{N}_T^d$, for $m = 1, \ldots, N$. Similarly, we define by $\{G^u_{0,t} : t \in \mathbb{N}_T^d\}$ the filtration generated by completion of the minimum $\sigma-$algebra $\mathcal{V}^{N}_{m=1} G^m_{0,t}$, $t \in \mathbb{N}_T^{d^2}$. For each $t \in \mathbb{N}_T^d$, let $\gamma_{t} \triangleq \left\{ (\tau, m) \in \{0, 1, \ldots, t\} \times \{1, 2, \ldots, N\} \right\}$. An information structure is the assignment to each $(t, k) \in \gamma_T$ of a data basis at time $t$, denoted by $\gamma_{t,k}$. For a given Borel measurable mapping $\gamma^k(t)$, the control actions at $k$th control station are
\[
u^k(t) = \gamma^k \left( \{y^m(t) : (\tau, m) \in \gamma_{t,k} \} \right), k \in \mathbb{Z}^N.
\]
(19)

We denote the set of admissible strategies at the $k$th control station at time $t \in \mathbb{N}_T^d$, by $\gamma^k(t) \in \mathbb{U}^k[t], t \sim t + 1$-tuple by
\[
\gamma_{[0,T]}^k \triangleq (\gamma_0^k(\cdot), \ldots, \gamma_{T}^k(\cdot)) \in \mathcal{U}^k[0, T] \triangleq \times_{k=0}^{T} \mathbb{U}^k[t],
\]
\[
\gamma_{[0,T]}^N \triangleq (\gamma_0^N(\cdot), \ldots, \gamma_{T}^N(\cdot)) \in \mathcal{U}(N)[0, T] \triangleq \times_{k=0}^{N} \mathbb{U}^k[0, T].
\]
Write \( \lambda_t(\cdot) \triangleq \prod_{m=1}^{N} \lambda_m^t(\cdot), t \in \mathbb{N}_0^T, \ y \triangleq \text{Vector}\{y_1, \ldots, y_N\}, \ h \triangleq \text{Vector}\{h_1, \ldots, h_N\}, \ D^\Delta \triangleq \text{diag}\{D_1^\Delta, \ldots, D^\Delta_N\} \). Consider the measurable functions
\[
f(t, \cdot) \triangleq x_0^t(t) = x_{k-1}^t(1), \quad k = 1, \ldots, N, \quad t \in \mathbb{N}_0^T-1,
\]
\[
h^m(t, \cdot) \triangleq h^m(t, x(t), u(t)) + D^m(t)h^m(t), \quad m = 1, \ldots, N.
\]
For any admissible \( u \equiv \gamma_{[0,T]} \in \mathbb{U}^{(N)}[0,T] \), under measure \( \mathbb{P}^u \) the pay-off is
\[
\mathbb{E}^u \left\{ \sum_{t=0}^{T-1} \ell(t, x(t), u^1(t), \ldots, u^N(t)) + \varphi(x(T)) \right\}.
\]

Conclusion III.1 The decentralized stochastic control optimization problem (24), (25) with pay-off (26) can be transformed to the equivalent static optimization problem with pay-off (21), where \( \{x(t), y(t) : t \in \mathbb{N}^T_N\} \) are independent sequences, distributed according to \( x(0) \sim \Pi_0(dx), x(t) \sim \zeta_t(\cdot), t \in \mathbb{N}^T, y^m(t) \sim \lambda^m_t(\cdot), t \in \mathbb{N}^T,m = 1, \ldots, N \).

Consequently, we can apply static team optimality in [8].

Theorem III.2 Suppose none of the control stations have perfect recall. Assume the following conditions hold.

1. \( L(x, y) \) is convex and differentiable uniformly in \((x, y) \in \mathbb{R}^{n+m}\).
2. \( \gamma_{[0,T]} \in \mathbb{U}^{(N)}[0,T] \) such that \( J(\gamma_{[0,T]}) < \infty \).
3. For all \( \gamma_{[0,T]} \in \mathbb{U}^{(N)}[0,T] \) such that \( J(\gamma_{[0,T]}) < \infty \),

The team problem is defined by
\[
\text{inf} \{ J(\gamma_{[0,T]}) : \gamma_{[0,T]} \in \mathbb{U}^{(N)}[0,T] \}.
\]

This is the transformed equivalent stochastic team problem.

Initial Decentralized Stochastic System: Discrete-Time-

(\( \Omega, \mathbb{F}, \{\mathbb{F}_t : t \in \mathbb{N}^T_N\}, \mathbb{P}^u \)). The initial stochastic control problem is introduced as follows. It can be shown that \( \{\Theta^u_t : t \in \mathbb{N}^T_N\} \) is an \( \Omega, \mathbb{F}, \{\mathbb{F}_t : t \in \mathbb{N}^T_N\}, \mathbb{P}^u \)-martingale, and hence \( \int \Theta^u_t d\mathbb{P} = 1 \). Therefore, we can define a probability measure \( \mathbb{P}^u \) on \( (\Omega, \mathbb{F}_t : t \in \mathbb{N}^T_N) \) by setting \( \frac{d\mathbb{P}^u}{d\mathbb{P}} |_{\mathbb{F}_t} = \Theta^u_t, t \in \mathbb{N}^T_N. \) Under measure \( \mathbb{P}^u \), the processes
\[
u^m(t) \triangleq G^{-1}(t-1) \left( x(t) - f(t-1, x^t_0, u^t_0) \right),
\]
\[
\gamma^m(t) \triangleq D^m(t) \left( y^m(t) - h^m(t, x(t), u(t)) \right),
\]
are two sequences of independent normally distributed RVs with densities, \( \zeta_t(\cdot), t \in \mathbb{N}^T, \) and \( \lambda^m_t(\cdot), t \in \mathbb{N}^T, m = 1, \ldots, N, \) respectively. Therefore, under measure \( \mathbb{P}^u \), the discrete-time stochastic system is (for \( m = 1, \ldots, N \))
\[
x(t+1) = f(t, x^t_0, u(t)) + G(t)w^m(t+1),
\]
\[
y^m(t) = h^m(t, x(t), u(t)) + D^m(t)\gamma^m(t).\]

(28) can be expressed under original measure \( \mathbb{P}^u \) via (27).
A. Witsenhausen Counterexample [6]

State Equations: \( x_1 = x_0 + u_1 \), \( x_2 = x_1 - u_2 \),
Output Equations: \( y_0 = x_0 \), \( y_1 = x_1 + ... \) of Control, Signals and Systems, p. 41, November 2013. [Online]. Available: http://arxiv.org/abs/1309.1913, pages ∼50

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Theorem IV.1 (Necessary Conditions for Team Optimality under Original Measure \( \mathbb{P}_u \))

Suppose Assumptions II.1 hold and \( K = M = N \).

For an element \( u^o \in U^{(N)}(0, T) \) with the corresponding solution \( x^o \in B_{\mathcal{F}T}(0, T), L^2(\Omega, \mathcal{R}^N) \) to be team optimal, it is necessary that the following hold.

1. There exists a square integrable semi martingale which satisfies the BSDEs (31), (32).

2. The following variational inequality is satisfied:

\[
\mathbb{E}^u \left\{ H(t, x^o(t), \psi(t), q_{11}^o(t), q_{22}^o(t), u_{\sim i}^o(t), u^i) \big| \mathcal{G}_t^i \right\}
\geq \mathbb{E}^u \left\{ H(t, x^o(t), \psi(t), q_{11}^o(t), q_{22}^o(t), u^o(t)) \big| \mathcal{G}_t^i \right\},
\forall u^i \in \mathcal{A}^i, a.e.t. \in [0, T], \mathbb{P}_u^{a} \big| \mathcal{G}_t^i = a.s., i \in \mathcal{Z}_N. \quad (33)
\]

Proof: The derivation is given in [13].

The necessary conditions for team optimality can be expressed under \((\Omega, \mathcal{F}, \{\mathcal{F}_t : t \in [0, T]\}, \mathbb{P}) \) [13], and these are projections of the Hamiltonian (33).

Existence of optimal strategies and sufficient team and Person-by-Person optimality conditions are given in [13].

Whether Girsanov’s measure transformation suffices as a basis to address existence/characterization questions of optimal decentralized decisions remains, however, to be seen.

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REFERENCES