Coordinates transformations in nonlinear time–delay systems

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Abstract—When considering nonlinear time delay systems, the definition of an invertible change of coordinates is no more granted. Starting from a set of \(n-1\) exact differentials independent over \(K[\delta]\) in the variables \(x(t), \cdots, x(t-sD)\), with \(D\) a constant commensurate delay and \(x\) an \(n\) dimensional variable, we give necessary and sufficient conditions for the existence of an exact one–form such that to get a basis for \(\text{span}_{K[\delta]}\{dx_{[0]}\}\).

I. INTRODUCTION

When dealing with nonlinear time delay systems, and their equivalence to particular structures, one major point that arises is the computation of change of coordinates which are bicausal, that is: not only the new state variables can be expressed as causal functions in the given state variables and their delays, but the converse has also to be true.

Consider for instance the system

\[
\begin{align*}
\dot{x}_1 &= 0 \\
\dot{x}_2 &= 0 \\
y &= x_1(t)x_1(t-1) + x_2(t)x_2(t-1)
\end{align*}
\]

there is no bicausal change of coordinates \((z_1(t), z_2(t))\) such that \(z_1 = x_1(t)x_1(t-1) + x_2(t)x_2(t-1)\) and consequently, there is no canonical decomposition of the system with respect to observability. This problem is formulated and solved in this paper.

In [2], through the introduction of the Extended Lie bracket operator, necessary and sufficient conditions were given, under which a given set of nonlinear one-forms in the \(n\)-dimensional delayed variables \(x(t), \cdots, x(t-sD)\), with \(D\) constant but unknown, and whose right annihilator is causal, are integrable. This result has been successfully used to study many properties of time delay systems and related to the accessibility and observability properties [1], [3], [4], [11].

However one major problem arises in these contexts and remains still open. Given a set of exact one–forms, find, if possible, a suitable completion to obtain a bicausal change of coordinates. In fact, to the best of our knowledge, the most general result stated on this topic can be found in [14] where it was shown that under some appropriate assumptions any basis of a closed submodule of rank \(k\) can be completed by one–forms to build a basis of the reference \(n\)-dimensional module \(M = \text{span}\{dx_1,...,dx_n\}\). This result is not valid in general when the one–forms are required to be exact. A typical counter-example is given by \(n = 2\), \(k = 1\) and \(\omega = d(x_1(t)x_1(t-1) + x_2(t)x_2(t-1))\). However, it is crucial to characterize the possibility for such a basis completion, when defining bicausal changes of coordinates.

In the present paper necessary and sufficient conditions are provided for solving the problem when starting with \(n-1\) exact differentials depending on an \(n\)-dimensional variable and its delays. More precisely in Section II, we introduce the necessary notation and recall some basic results. In Section III the main result is given whereas an algorithm for the computation of the searched exact differential is given in Section IV. An example shows the technical details in Section V. Finally, some concluding remarks are given in Section VI.

II. PRELIMINARIES AND NOTATIONS

In the following, functions are considered in the variables \(x(t) \in \mathbb{R}^n\) as well as in their delays \(x(t-kD)\), where \(D\) represents a constant, possibly unknown, delay. That is, functions are considered of the form \(\lambda(x(t), x(t-D), \cdots, x(t-sD))\).

We will denote by \(x_{[n]}^T = (x^T(t), \cdots, x^T(t-sD)) \in \mathbb{R}^{(s+1)n}\), with \(x_{[0]} = [x_{1,[0]}, \cdots, x_{n,[0]}]^T = x(t) \in \mathbb{R}^n\), the instantaneous values of the variables. Moreover \(x_{[s]}^T(-p) = (x^T(t-pD), \cdots, x^T(t-sD-pD))\). When no confusion is possible the subindex will be omitted so that \(x\) will stand for \(x_{[n]}\) and \(x(-p)\) will stand for \(x_{[s]}(-p)\).

The following notation taken from [2], [9], [16] will be used:

1. \(K\) denotes the field of meromorphic functions of a finite number of variables in \(\{x_{[0]}(-i), i \in \mathbb{N}\}\);
2. Given a function \(f(x_{[s]})\), we will denote by

\[\text{K}\]
\[ f(-l) = f(x_{[s]}(-l)); \]

- \( d \) is the standard differential operator;
- \( \delta \) represents the backward time-shift operator: for \( \alpha(x), f(x) \in \mathcal{K} \):
  \[
  \delta[\alpha(x) df(x)] = \alpha(x(-1)) df(x) \\
  = \alpha(x(-1)) df(x(-1));
  \]
- \( \mathcal{K} (\delta) \) is the (left) ring of polynomials in \( \delta \) with coefficients in \( \mathcal{K} \). Every element of \( \mathcal{K} (\delta) \) may be written as \( \alpha(\delta) = \sum_{j=0}^{r_\alpha} \alpha_j(x) \delta^j \), with \( \alpha_j(\cdot) \in \mathcal{K} \) and \( r_\alpha = \deg(\alpha(\delta)) \) the polynomial degree in \( \delta \). By convention \( \alpha_i(\cdot) = 0 \) for \( i > r_\alpha \). Let \( \beta(x) = \sum_{j=0}^{r_\beta} \beta_j(x) \delta^j \) be an element of \( \mathcal{K} (\delta) \) of polynomial degree \( r_\beta \) and set again \( \beta_j(\cdot) = 0 \) for \( j > r_\beta \). Then addition and multiplication on this ring are defined by [16]
  \[
  \alpha(\delta) + \beta(\delta) = \sum_{i=0}^{\max(r_\alpha, r_\beta)} (\alpha_i(x) + \beta_i(x)) \delta^i \\
  \alpha(\delta) \beta(\delta) = \sum_{i=0}^{r_\alpha} \sum_{j=0}^{r_\beta} \alpha_i(x) \beta_j(x(-i)) \delta^{i+j}
  \]
- Let \( \tau_i(x_{[j]}) \), for \( i \in [1, j] \), be vector fields defined on an open set \( \Omega \subseteq \mathbb{R}^{n(l+1)} \). Then
  \[
  \Delta = \text{span}\{\tau_i(x_{[j]}), i = 1, ..., j\}
  \]
represents the distribution generated by the vector fields \( \tau_i(\cdot) \) and defined in \( \mathbb{R}^{n(l+1)} \). \( \Delta \) represents its involutive closure, that is, for any two vector fields \( \tau_i(\cdot), \tau_j(\cdot) \in \Delta \) then also the Lie bracket \([\tau_i, \tau_j] = \tau_i\tau_j - \tau_j\tau_i \in \Delta \). Instead let \( \tau_i(x_{[j]}, \delta) \) be \( \mathcal{K}^n(\delta) \)-valued functions over \( \mathcal{K} (\delta) \), that is any \( \tau(x, \delta) \in \Delta(\delta) \) can be expressed as \( \tau(x, \delta) = \sum_{i=1}^{j} \tau_i(x, \delta) \alpha_i(x, \delta) \).

We recall hereafter the definitions of Extended Lie Bracket and Extended Lie derivative introduced in [2], [3], where they were successfully used to tackle the integrability problem of one–forms. To this end consider \( r(x, \delta) = \sum_{j=0}^{s} r^j(x) \delta^j \), and set \( r^{s+j}(x) = 0 \) for any \( j > 0 \).

**Definition 1:** Given the function \( \tau(x_{[j]}) \) and the element \( r(x, \delta) = \sum_{j=0}^{s} r^j(x) \delta^j \), the Extended Lie derivative \( L_{r_{[j]}}(x) \tau(x_{[j]}) \) is defined as

\[
L_{r_{[j]}}(x) \tau(x_{[j]}) = \sum_{j=0}^{s} \frac{\partial \tau(x_{[j]})}{\partial x_0} r^{s-j}(x(-l)) \tau(x_{[j]}(-l)) \quad (1)
\]

**Definition 2:** Let \( r_i(x, \delta) = \sum_{j=0}^{s} r^j_i(x) \delta^j \), \( i = 1, 2 \). For any \( k, l \geq 0 \), the Extended Lie bracket \([r^1_1(\cdot), r^2_1(\cdot)]\) on \( \mathbb{R}^{n(l+1)} \), is defined as

\[
[r^1_1(\cdot), r^2_1(\cdot)]_{E_0} = \left( L_{r^1_1(x)} r^2_1(x) - L_{r^2_1(x)} r^1_1(x) \right) \frac{\partial}{\partial x_0},
\]

\[
[r^1_1(\cdot), r^2_1(\cdot)]_{E_j} = \sum_{j=0}^{\min(k+1, l)} \left( [r^1_{k-j}(\cdot), r^2_{l-j}(\cdot)]_{E_0} \right) \frac{\partial}{\partial x_0} \quad (3)
\]

Definitions 1 and 2 are consistent with the definitions of Lie derivative and Lie bracket in the delay–free case.

Finally let us recall the following result which states under which conditions given two submodules \( A \) and \( B \) of \( \mathcal{M} \) such that \( A \subset B \) then starting from the basis of \( A \) it is possible to compute a basis completion in \( B \).

**Theorem 1:** [14] Let \( L_1 \) and \( L_2 \) be two matrices with entries in \( \mathcal{K}(\delta) \), and let \( A = L_1^+ \) and \( B = L_2^+ \) be two submodules of \( \mathcal{M} \), with \( A \subset B \). Suppose that \( \{\omega_i, i = 1, 2, \ldots, r_A\} \) form a basis for \( A \). Then there exists vectors \( \omega_j, j = r_A + 1, \ldots, r_B \) such that the extended set \( \{\omega_i, i = 1, 2, \ldots, r_B\} \) is a basis for \( B \).

However if the basis of \( A \) is constituted by exact differentials, then it is not proven that one can complete the basis with exact differentials. This is the topic of next section in the special case where \( \text{rank}(B) = \text{rank}(A) + 1 \).

### III. Main result

Hereafter we will show that given \( n-1 \) functions \( \lambda_i(\lambda_{[n]}) \), \( i \in [1, n-1] \) whose differentials are independent over \( \mathcal{K}(\delta) \), and such that

\[ \text{span}_{\mathcal{K}(\delta)} \{d\lambda_1, \cdots, d\lambda_{n-1}\} \]

is closed and its right annihilator is causal, it is always possible to compute a basis completion for \( \mathbb{R}^n \). The following result can be stated.

**Theorem 2:** Given \( n-1 \) functions \( \lambda_i(\lambda_{[n]}) \), \( i \in [1, n-1] \), whose differentials are independent over \( \mathcal{K}(\delta) \), then there
exists a function $\theta(x^{\bar{\alpha}})$ such that
\[
\begin{pmatrix}
  d\lambda_1 \\
  \vdots \\
  d\lambda_{n-1} \\
  d\theta
\end{pmatrix} = T(x, \delta) dx^{[0]}
\]
with $T(x, \delta)$ unimodular if and only if

i) $\text{span}_{K(\delta)}\{d\lambda_1, \ldots, d\lambda_{n-1}\}$ is closed

ii) its right annihilator is causal modulo post multiplication by a nonzero coefficient

As a consequence $dz^{[0]} = T(x, \delta) dx^{[0]}$ defines a bicausal change of coordinates.

**Proof:** Necessity: assume that there exists a $d\theta$ which forms a basis completion over $\mathbb{R}^n$ but i) is not satisfied. Then there exist an $\omega(x, \delta) dx^{[0]}$ and an $\alpha(x, \delta) \neq 0$, such that

\[
\omega(x, \delta) dx^{[0]} \not\in \text{span}_{K(\delta)}\{d\lambda_1, \ldots, d\lambda_{n-1}\}
\]

\[
\alpha(x, \delta) \omega(x, \delta) dx^{[0]} \in \text{span}_{K(\delta)}\{d\lambda_1, \ldots, d\lambda_{n-1}\}.
\]

On the other hand, by assumption,

\[
\omega(x, \delta) dx^{[0]} \in \text{span}_{K(\delta)}\{d\lambda_1, \ldots, d\lambda_{n-1}, d\theta\}
\]

which means that there exist coefficients $\beta_j(x, \delta)$, $j \in [1, n-1]$, $\gamma(x, \delta) \neq 0$, such that

\[
\omega(x, \delta) dx^{[0]} = \sum_{j=1}^{n-1} \beta_j(x, \delta) d\lambda_j(x) + \gamma(x, \delta) d\theta
\]

Since

\[
\alpha(x, \delta) \omega(x, \delta) dx^{[0]} = \sum_{j=1}^{n-1} \beta_j(x, \delta) d\lambda_j(x)
\]

\[
\alpha(x, \delta) = \alpha(x, \delta) \sum_{j=1}^{n-1} \beta_j(x, \delta) d\lambda_j(x) + \alpha(x, \delta) \gamma(x, \delta) d\theta
\]

this immediately implies that

\[
\alpha(x, \delta) \gamma(x, \delta) = 0
\]

which yields a contradiction since by assumption $\alpha(x, \delta) \neq 0$ and $\gamma(x, \delta) \neq 0$.

As for ii), since $T(x, \delta)$ is unimodular, also its inverse $T^{-1}(x, \delta)$ is unimodular. The last column of $T^{-1}(x, \delta)$ constitutes the right annihilator of $\text{span}_{K(\delta)}\{d\lambda_1, \ldots, d\lambda_{n-1}\}$ and is by construction causal.

**Sufficiency:** Since i) and ii) are satisfied, according to Theorem 1, there exists an $\omega(x, \delta) dx^{[0]}$, which can be computed using Smith decomposition, such that denoting by $d\lambda = (d\lambda_1^T, \ldots, d\lambda_{n-1}^T)^T$, then

\[
\left(\begin{array}{c}
  d\lambda(x) \\
  \omega(x, \delta) dx^{[0]}
\end{array}\right) = T(x, \delta) dx^{[0]}
\]

is characterized by a unimodular matrix $T(x, \delta)$. Let $T^{-1}(x, \delta) = (p_1(x, \delta), \ldots, p_n(x, \delta))$, set $\bar{p} = p_n$ and accordingly by $\bar{p}(x, \delta) = \sum_{i=0}^{\infty} \bar{p}(\delta)$. Consider now the sequence of distributions

\[
\Delta_0 = \text{span} \left( \begin{array}{cccccc}
  \bar{p}^0 & \bar{p}^1 & \cdots & \bar{p}^s & 0 & I \\
  0 & 1 & \cdots & 0 & 1 & \\
  \vdots & \ddots & \ddots & \vdots & \ddots & \\
  \vdots & \cdots & \cdots & \cdots & \cdots & \\
  0 & \cdots & \cdots & \cdots & \cdots & 1 \\
  \end{array}\right)
\]

for $\ell \geq 1$ and denote now by $\Delta_{\ell, 0}$ the following distribution

\[
\Delta_{\ell, 0} = \text{span} \left( \begin{array}{cccccc}
  \bar{l}^1 & \cdots & \bar{l}^s & 0 & \\
  \bar{l}^0(-1) & \cdots & \bar{l}^{s-1}(-1) & \cdots & \\
  0 & \cdots & \cdots & \cdots & \\
  \vdots & \cdots & \cdots & \cdots & \cdots & \\
  0 & \cdots & \cdots & \cdots & \cdots & 1 \\
  \end{array}\right)
\]

By assumption $d\lambda_i(x) = \Lambda_i(x, \delta) dx^{[0]}$ for $i \in [1, n-1]$ and $\omega(x, \delta)$ satisfy

\[
\Lambda_i(x, \delta) \bar{p}(x, \delta) = 0, \quad i \in [1, n-1]
\]

\[
\omega(x, \delta) \bar{p}(x, \delta) = \omega_0 \bar{p}_0 = 1
\]

Let $j$ be the maximum between the degree of $\Lambda_i(x, \delta)$ $i \in [1, n-1]$ and the degree of $\omega(x, \delta)$, then we have that $d\lambda_j \Delta_j = 0$, whereas $[\omega_0, \omega_1, \ldots, \omega_j] \Delta_j \neq 0$ since by assumption $\omega_0 \bar{p}^0 = 1$, whereas $[\omega_0, \omega_1, \ldots, \omega_j] \Delta_j 0 = 0$.

Assume now that one cannot choose $\omega(x, \delta) dx^{[0]}$ to be an exact differential, that is one cannot find $\omega(x, \delta) dx^{[0]} = d\theta$. Since $\omega(x, \delta) \perp \Delta_{j, 0}$, this would imply that there exists an Extended Lie Bracket such that $[\bar{p}^i, \bar{p}^k]_{E,k} = \tau_0(x) + \bar{p}^j(x) \alpha_0(x) \frac{\theta^i}{\theta^{j+1}}$, (with $i \geq k$) where $\tau_0(x) = \sum_{i=0}^{\infty} \tau_0^{(-l)}(\bar{p}) \frac{\theta^i}{\theta^{j+1}} \in \Delta_{j, 0}$, $\alpha_0 \neq 0$.

Consider now $\Delta_{j+1}$ and correspondingly $\Delta_{j+1, 0}$. Accordingly consider the Extended Lie Bracket $[\bar{p}^{j+1}, \bar{p}^{k+1}]_{E,k+1}$. 477
By construction

\[ \pi_{k+1} = \frac{\partial}{\partial x_0} \left( \pi'_0(x) + \pi'_1(x) \alpha_0(x(-1)) \right) + \cdots \]

where \( \pi_1(x) = \sum_{l=0}^{\mu+1-l} \pi^{\mu+1-l}(x(-l)) \frac{\partial}{\partial x_0} \in \Delta_{j+1,0}. \)

In fact, if (7) were not satisfied then

\[ \sum_{l=0}^{\mu+1} \pi^{\mu+1-l}(x(-l)) \frac{\partial}{\partial x_0} \in \Delta_{j+1,0}. \]

against the assumption of \( n-1 \) independent exact differentials in the left kernel of \( \bar{p}(x, \delta) \). Iterating the reasoning one gets that for any \( \beta \geq 0, \)

\[ [p^{i+\beta}, p^{k+\beta}]_{E,k+\beta} = \tau_\beta(x) \]

and \( \tau_\beta(x) = \sum_{l=0}^{\mu+1-l} x^{\mu+1-l}(x(-l)) \frac{\partial}{\partial x_0} \in \Delta_{j+1,0}. \)

Since by assumption \( \bar{p} = 0 \) for \( l > s \), one gets that as soon as \( k + \beta \geq s + 1, [p^{i+\beta}, p^{k+\beta}]_{E,k+1} = 0. \) Furthermore since \( \tau_0 \in \Delta_{j,0}, \) then necessarily there exists an index \( l \leq s \) such that \( \tau^{l+1} = 0, \forall i \geq 0. \) As a consequence, there exists an index \( \beta \leq s, \) such that for any \( \nu \in [\beta, s] \)

\[ 0 = [p^{i+\nu}, p^{k+\nu}]_{E,0} = \sum_{l=0}^{\nu} \bar{p}^{\nu-l}(x) \alpha_l(x(-\nu + \gamma)) \frac{\partial}{\partial x_0}. \]

Consider now the distribution \( \bar{\Delta}_j \) obtained by combining linearly the columns of \( \Delta_j \) through the matrix

\[ \alpha = \begin{pmatrix} \alpha_0 & \alpha_1 & \cdots & \alpha_s & \cdots & 0 \\ 0 & \alpha_0(-1) & \cdots & \cdots & \cdots \\ 0 & \cdots & \alpha_0(-j) & \cdots & \alpha_s(-j) & 0 \\ 0 & \cdots & \cdots & \cdots & 0 & I \end{pmatrix} \]

that is (dropping the word "span")

\[ \bar{\Delta}_j = \begin{pmatrix} p^0 & \bar{p}^1 & \cdots & \bar{p}^s & \cdots & 0 \\ 0 & \bar{p}^0(-1) & \cdots & \cdots & \cdots \\ \vdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix} \]

\[ \begin{pmatrix} \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix} \]

By construction we thus have that

\[ \bar{\Delta}_j = \begin{pmatrix} p^0 & \bar{p}^1 & \cdots & \bar{p}^s & \cdots & 0 \\ 0 & \bar{p}^0(-1) & \cdots & \cdots & \cdots \\ \vdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix} \]

\[ \begin{pmatrix} \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix} \]

IV. COMPUTATION OF THE CHANGE OF COORDINATES

If the conditions of Theorem 2 are satisfied, then one has to compute the missing exact differential which enables to define the desired change of coordinates. To this end the proofs of Theorem 1 and Theorem 2 give also an insight on the procedure to follow and which is pointed out hereafter.

\begin{itemize}
  \item Step 1 Starting from the given set of exact differentials
  \[ \begin{pmatrix} d\lambda_1 \\ \vdots \\ d\lambda_{n-1} \end{pmatrix} = \Lambda(x, \delta)dx_0 \]
\end{itemize}
apply Smith decomposition on the matrix $\Lambda(x, \delta)$ which by assumption has rank $n - 1$. As a consequence

$$\Lambda(x, \delta) = [\Gamma(x), 0] S(x, \delta)$$

where $\Gamma(x)$ is a $n - 1$ full rank square matrix and $S(x, \delta)$ is an unimodular matrix of dimension $n$.

- **Step 2** Pick the last row of $S(x, \delta)$, denoted by $s_n(x, \delta)$.

- **Step 3** The searched exact differential $d\theta = \alpha_0 s_n(x, \delta) dx_0$ with $\alpha(x)$ a scalar non zero function.

V. AN EXAMPLE

Consider $\lambda = x_2 + x_1(1)x_2(2) - x_2(1)$ and compute its differential which is given by

$$d\lambda = (x_2(-2)\delta + x_1(-1)\delta^2 - \delta) dx_0$$

Let us check the conditions of Theorem 2.

Condition i) is satisfied since $\text{span}_{K(\delta)} \{d\lambda\}$ is closed. In order to check Condition ii) we use Smith decomposition to get $s_n(x, \delta) dx_0$. For $x_2(-i) \neq 0$ for $i \in \{0, 3\}$ we have that

$$d\lambda = (x_2(-2)\delta + x_1(-1)\delta^2 - \delta) dx_0$$

$$= (x_2(-1) - 0) \begin{pmatrix} \delta x_2(-1) & 1 \\ 1 & 0 \end{pmatrix} \times \begin{pmatrix} 1 & x_1(-1) \\ 0 & x_2(-1) \end{pmatrix}$$

so that

$$s_n(x, \delta) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \delta x_2(-1) & 1 \\ \frac{1}{x_2(-1)} & 0 \end{pmatrix} \begin{pmatrix} 1 & x_1(-1) \\ 0 & x_2(-1) \end{pmatrix}$$

and thus the right annihilator of $d\lambda(x)$ is causal and is given by

$$p(x, \delta) = \begin{pmatrix} -1 \\ \delta \end{pmatrix} (1 - \delta + x_1(1)^2) x_2(-1)$$

so that condition ii) is satisfied.

In order to compute the desired basis completion, let us note that since we are considering the case of a single one-form, then as shown in [14], $\Omega = s_n(x, \delta) dx_0$ is integrable if and only if, $d\Omega \wedge \Omega = 0$ where $\wedge$ denotes the exterior product. We have that since $\Omega = dx_1 + \frac{x_1}{x_2(1)} dx_2(-1) - \frac{1}{x_2(1)} dx_2$

$$d\Omega = \frac{1}{x_2(-1)} dx_1 \wedge dx_2(-1) + \frac{1}{x_2^2(-1)} dx_2(-1) \wedge dx_2$$

so that

$$d\Omega \wedge \Omega = - \frac{1}{x_2^2(-1)} dx_1 \wedge dx_2(-1) \wedge dx_2 = 0$$

It is easily seen that the searched exact form is thus given by $d\theta = \alpha_0 = dx_2 - d(x_1 x_2(-1))$.

VI. CONCLUSION

In the present paper necessary and sufficient conditions were given such that a set of given $n-1$ functions in the state variable $x(t)$ and its delays $x(t-iD)$, $i \in \{1, s\}$, can be used to define a bicausal change of coordinates. The computation of the missing exact differential which completes the basis, can be easily carried out using a procedure detailed in the paper. Such a result represents an important first step to address the general case, which however is not trivial. It is sufficient to recall that, as already noted in [14], given two one-forms $\Omega_1$ and $\Omega_2$ the conditions $d\Omega_i \wedge \Omega_1 \wedge \Omega_2 = 0$ for $i \in \{1, 2\}$ are no more necessary to ensure the integrability of $(\Omega_1, \Omega_2)$.

REFERENCES


