A Set Stabilization Approach for Circular Formations of Rigid Bodies

Mohamed I. El-Hawwary*

Abstract—The paper presents a set stabilizing framework for distributed control design to achieve circular formations for rigid bodies in three-dimensional Euclidean space. Information exchange between the rigid bodies is modeled by a directed graph which is assumed to have a spanning tree. Actuation scenarios of fully actuated, degree-one underactuated, and degree-two underactuated rigid bodies are addressed. Control design is based on a hierarchical approach that relies on a reduction principle for asymptotic stability of closed sets.

I. INTRODUCTION

In [1], formation control was addressed in the context of set stabilization. A hierarchical approach was introduced for solving a formation control problem for dynamic unicycles. The formation problem was formulated as a set stabilization problem with the desired formation modeled as a goal set in the unicycles state space. The problem was then broken down into three simpler decoupled sub-problems of stabilizing three nested sets, the smallest of which being the goal set. The solution of the formation problem followed from that of the sub-problems utilizing a recently developed reduction principle for asymptotic stability of closed sets [2], [3].

In this paper, the set stabilizing view point introduced in [1] for formation control is further investigated. A formation problem is addressed for a system of rigid bodies in three dimension. The bodies are assumed to be either fully actuated, degree-one underactuated (2 thrusters - 3 torques), or degree-two underactuated (1 thrust - 3 torques). In formation, the rigid bodies are required to follow circular paths with the same orientation and radius, and with each body forward direction tangent to its path. The formation is specified by arbitrary displacements of the centers of the circular paths, and arbitrary relative headings of the rigid bodies. Like in [1], the formation control problem is broken down into a number of decoupled sub-problems of stabilizing nested sets, and the solution follows from the reduction principle of [2], [3].

The problem addressed here is of interest in multi-agent studies. Circular formations have been investigated in collective behaviour studies, see for instance [4], [5], and the problem is of potential interest in formation control applications, especially in Aerospace. Geometric formations, especially cyclic strategies, received wide interest in that field [6]. For example, in [7] cyclic pursuit strategies were studied for single- and double-integrators in three dimension. Among the formations achieved was evenly-spaced circular formations, an instance of the problem addressed here. Potential applications were discussed, and experimental results were provided. Note that, the model used in this paper is closer to what vehicles in three dimension, such as satellites and aerial and underwater vehicles, would be modeled by.

Notation. For scalars $a_1, \ldots, a_n$, $\text{diag}(a_1, \ldots, a_n)$ denotes the diagonal matrix with diagonal entries $a_i$. If $A, B$ are two matrices, $\text{diag}(A, B)$ denotes the block diagonal matrix with blocks $A, B$; and $A \otimes B$ denotes their Kronecker product. The index set $\{1, \ldots, n\}$ is denoted by $n$, and the $n$-vector of ones is denoted by $1$. For $x \in \mathbb{R}$, $x \bmod 2\pi$ denotes its value modulo $2\pi$, and if $\theta, x \in \mathbb{R}$, then $x = \theta \bmod 2\pi$ states that $x \in \{\theta + 2\pi k, k \in \mathbb{Z}\}$. Similarly, if $\theta, x \in \mathbb{R}^n$, then $x = \theta \bmod 2\pi$ states that $x_i = \theta_i \bmod 2\pi, i = 1, \ldots, n$. For any two vectors $u, v \in \mathbb{R}^3$, $\angle(u, v)$ denotes the angle that $u$ makes relative to $v$. $\text{Sat}(\mathbb{R})$ will be used to denote the class of $C^1$ saturation functions $\varphi : \mathbb{R} \to \mathbb{R}$ such that for all $y \in \mathbb{R}$, $\varphi(y)y \geq 0$, $|\varphi(y)| < 1$, and $\varphi(y) = 0$ only when $y = 0$. For any dynamical system $\dot{x} = f(t, x)$, $x(t)$ will denote its solution with initial condition $x(0)$. Finally, the vectors $e_1, e_2, e_3$ denote the standard basis in $\mathbb{R}^3$; and, for $x \in \mathbb{R}^3$, $x^\times$ denotes its skew symmetric matrix representation where $x^\times y = x \times y$ for any $y \in \mathbb{R}^3$.

II. PROBLEM FORMULATION

A system of $n \geq 2$ rigid bodies in three-dimensional Euclidean space is considered. Rigid body $i$, has a body frame $B_i$ attached to it, and its attitude is parametrized by a rotation matrix $B_i \in \text{SO}(3)$ where $b_j^i = B_i e_j, j = 1, 2, 3$. It is assumed that each rigid body is provided with three thrusters $f_1^i, f_2^i, f_3^i$ in the directions $b_1^i, b_2^i, b_3^i$ and three torques $\tau_1^i, \tau_2^i, \tau_3^i$ around the body axes. The following states are defined: $x^i, \dot{x}^i \in \mathbb{R}^3$ are the $i$-th rigid body center of mass and linear velocity in inertial frame, $B_i \in \text{SO}(3)$ is the $i$-th rigid body attitude, and $\omega^i \in \mathbb{R}^3$ is its angular velocity in frame $B_i$.

The state vector of the $i$-th rigid body is given by $\chi_i = (x^i, \dot{x}^i, B_i, \omega^i) \in \mathcal{X}_i := \mathbb{R}^3 \times \mathbb{R}^3 \times \text{SO}(3) \times \mathbb{R}^3$, and the collective state vector is given by $\chi = (\chi_1, \ldots, \chi_n) \in \mathcal{X} := \mathcal{X}_1 \times \cdots \times \mathcal{X}_n$. The system is modeled by

$$
\begin{align*}
m_i \ddot{x}^i &= B_i \dot{f}^i, & \dot{B}_i &= B_i (\omega^i)^\times, \\
\omega^i &= \mathbb{I}^{-1}_3 (\tau^i - (\omega^i)^\times \mathbb{I}_3) , & i \in \mathbb{n},
\end{align*}
$$

where $f^i := [f_1^i \ f_2^i \ f_3^i]^\top$, $\tau^i := [\tau_1^i \ \tau_2^i \ \tau_3^i]^\top$, $m_i$ is the mass of rigid body $i$, and $\mathbb{I}_3$ is its inertia matrix in $B_i$. Let $x = (x^1, \ldots, x^n)$, $B = (B_1, \ldots, B_n)$, and $b_1 = (b_1^1, \ldots, b_1^n)$.

The information exchange between the rigid bodies shall be modeled by a directed graph $G$. An edge in $G$ from node $i$ to node $j$ indicates that body $i$ has access to certain states of body $j$. Information exchange between the rigid bodies is subject to the following conditions.

C1. Rigid body $i$ has access to its own orientation $B_i$, and

* PhD - Edward S. Rogers Sr. Department of Electrical and Computer Engineering, University of Toronto, Toronto M5S 3G4, Ontario, Canada. Email: melhawwary@control.utoronto.ca.
its linear and angular velocities \( \dot{x}_i, \omega_i \).

**C2.** Body \( i \) has access to relative orientations of rigid bodies visible to it according to \( \mathcal{G} \), and relative positions, linear and angular velocities, expressed in \( B_i \), of these bodies.

In this paper \( \mathcal{G} \) will be considered to be static and balanced, with a globally reachable node [8]. In the sequel, \( N(i) \) will be used to denote the set of nodes of \( \mathcal{G} \) connected to node \( i \), and \( L^i \) will be used to denote the Laplacian of \( \mathcal{G} \) with \( L^i \) its \( i \)-th row, and \( L_{(3)} := L \otimes I_3, L^i_{(3)} := L^i \otimes I_3 \).

**A. Circular formations problem for rigid bodies - CFPR**

For the \( n \) rigid bodies in (1) with sensor digraph \( \mathcal{G} \), design distributed feedbacks satisfying the following specifications:

(i) **Circular path following.** For a unit vector \( a \in \mathbb{R}^3 \), and a suitable set of initial conditions, each rigid body should follow a circular path of radius \( r > 0 \), whose plane is orthogonal to \( a \), and whose center is stationary but dependent on initial conditions. While body \( i \) follows its path, \( b^1_i \) should be in the direction of \( \dot{x}_i \). The bodies traverse the paths in a desired direction (clockwise or counter-clockwise) with respect to \( a \). At steady-state, all rigid bodies should have forward speed \( v > 0 \).

(ii) **Centers stabilization.** The rigid bodies should converge to a formation given by desired relative positions of the centers of the paths in specification (i).

(iii) **orientations stabilization.** At steady state, the rigid bodies should satisfy desired relative orientations given by desired differences of the headings \( b^1_i \)'s.

**B. CFPR as a set stabilization problem**

First, notice that specification (i) is equivalent to making

\[
\mathcal{C} = \{ \chi \in \mathcal{X} : a \cdot b^1_i = 0, \omega^2_i b^2_i + \omega^3_i b^3_i = \frac{v}{r} a, \dot{x}_i = v b^1_i, i \in \mathfrak{n} \}
\]

attractive \(^2\). Let

\[
x^i_c(\chi^i) = \dot{x}_i + ra \times b^1_i,
\]

and \( x^c(\chi) = [x^1_c(\chi^1)^\top \cdots x^n_c(\chi^n)^\top]^\top \). \( x^i_c(\chi^i) \) is the center of the circular path in specification (i), refer to Figure 1.

\(^1\)The feedbacks designed in this paper provide circular path following in counter-clockwise direction around \( a \), and can be modified easily to provide clockwise path following.

\(^2\)Definitions of stability and attractivity of sets used in this paper, and their local and relative counterparts, can be found in [2], [3].

To specify the formation in specification (ii), one way is to specify vectors \( p_1, \ldots, p_{n-1} \in \mathbb{R}^3 \), and enforce the relations \( x^i_c - x^{i+1}_c = p_i, i = 1, \ldots, n-1 \), refer to Figure 1. Using this, specification (ii) becomes equivalent to making the set \( \{ \chi \in \mathcal{C} : x^i_c - x^{i+1}_c = p_i, i = 1, \ldots, n-1 \} \) attractive. Let \( \alpha_i = 0, \alpha_i = \sum_{j=i}^{n-1} \beta_j, i = 1, \ldots, n-1 \), and \( \alpha = [\alpha_1 \cdots \alpha_n]^\top \).

Specification (ii) is then equivalent to making the set

\[
\mathcal{F} = \{ \chi \in \mathcal{C} : L_{(3)}(x^c(\chi) - \alpha) = 0 \}
\]

attractive. \( \alpha \) defines a formation modular a translation in \( \mathbb{R}^3 \).

For specification (iii), let \( a^1 \) be a unit vector perpendicular to \( a \), and let \( b_1 = [b^1_1 \cdots b^1_n]^\top \) where \( b^1_i = \angle(b^1_i, a^1) \).

To specify the desired relative orientations in specification (iii), one way is to specify desired angles \( \theta_1, \ldots, \theta_{n-1} \), and enforce the relations \( b^1_i - b_i^{1+1} = \theta_i, i = 1, \ldots, n-1 \), refer to Figure 1. Let \( \beta_0 = 0, \beta_i = \sum_{j=1}^{n-1} \theta_j, i = 1, \ldots, n-1 \), and \( \beta = [\beta_1 \cdots \beta_n]^\top \). Using this, it then follows that specification (iii) is equivalent to making the set \( \Gamma = \{ \chi \in \mathcal{F} : L(b_1 - \beta) = 0 \text{ mod } 2\pi \} \) attractive. Note that \( \beta \) defines unique relative headings up to circular rotation.

By the previous development, meeting specifications (i), (ii), and (iii) of CFPR simultaneously is equivalent to making

\[
\mathcal{F} = \{ \chi \in \mathcal{X} : a \cdot b^1_i = 0, \dot{x}_i = v b^1_i, \omega^2_i b^2_i + \omega^3_i b^3_i = \frac{v}{r} a, \ L_{(3)}(x^c(\chi) - \alpha) = 0, L(b_1 - \beta) = 0 \text{ mod } 2\pi, i \in \mathfrak{n} \}
\]

attractive. Adding to this a stability requirement, which is more practically desirable, CFPR will be seen as the requirement to asymptotically stabilize the goal set \( \Gamma \).

**C. Solution approach**

The approach used next for solving CFPR is hierarchical in nature. The solution is broken down into that of \( l \) sub-problems addressing the stability of nested sets \( \Gamma_1 \supseteq \Gamma_2 \supseteq \cdots \supseteq \Gamma_l \), with \( \Gamma_i \) being the goal set. The number \( l \) is different for the different actuation scenarios. For \( i \in \{1, \ldots, l\} \), sub-problem \( i \) is that of designing feedbacks to asymptotically stabilize \( \Gamma_i \) relative to \( \Gamma_{i-1} \) (\( \Gamma_0 := \mathcal{X} \)) for the closed-loop system. The feedbacks \( f^i(\chi), \tau^i(\chi), i \in \mathfrak{n} \), are hence designed to render \( \Gamma_i \) asymptotically stable relative to \( \Gamma_{i-1} \), \( i = 1, \ldots, l \). The question then becomes whether this implies that the goal set \( \Gamma_l \) is asymptotically stable.

The answer to the question above relies on what is called reduction principles. These principles have been studied for the stability of compact sets in [9], and, recently, for non-compact sets in [2], [3]. Using these principles, a hierarchical control design problem was solved in [3] bearing the answer to the previous question, refer to Proposition 14 in [3].

**III. CASE A: FULLY ACTUATED RIGID BODIES**

In this section CFPR is solved under the assumption that (1) is fully actuated. The problem is broken down into the following four sub-problems.

**SPA1.** Design feedbacks \( f^i(\chi), \tau^i(\chi), i \in \mathfrak{n} \), stabilizing a desired “kinematic behaviour” expressed by the set

\[
\Gamma_{A1} = \{ \chi \in \mathcal{X} : \dot{x}_i = u_i(\chi)b^1_i, \omega = w^i(\chi), i \in \mathfrak{n} \},
\]

where \( u_i(\chi), w^i(\chi) \) are \( C^1 \) functions to be defined in the following steps. On \( \Gamma_{A1} \), the rigid bodies take the form of
steered kinematic particles with inputs $u_i(\chi), w_i(\chi)$.

**SPA2.** For the kinematic motion on $\Gamma^A_1$, make the rigid bodies approach planes orthogonal to $a$, with separations consistent with the desired formation. This corresponds to stabilizing the set $\Gamma^A_2 \subset \Gamma^A_1$, where
$$
\Gamma^A_2 = \{ \chi \in \Gamma^A_1 : L(x_a(\chi) - \alpha_0) = 0, a \cdot b^1_i = 0, i \in \mathbf{n} \},
$$
where $x_a(\chi) = [x^1_a \cdots x^n_a]^\top$, $x^i_a = x^i \cdot a$, $i \in \mathbf{n}$, and $a_\alpha = [a_1 \cdot a \cdots a_n \cdot a]^\top$.

**SPA3.** On the planes in SPA2, make the rigid bodies approach the formation in specification (ii) of CFPR. This corresponds to stabilizing $\{ \chi \in \Gamma^A_2 : L(\chi)(x_c(\chi) - \alpha) = 0 \}$. Consider the centers of rotation in (2). Define their projections orthogonal to $a$ by $x^i_p = x^i - (x^i_c \cdot a) a$, $i \in \mathbf{n}$, and denote $x_p(\chi) = [(x^1_p)^\top \cdots (x^n_p)^\top]^\top$. Also, define the projection of $\alpha$ orthogonal to $a$ by $a_{\alpha} = [a_1^\top \cdots a_n^\top]^\top$, where $a_i = a_1 \cdot a$. Using this, SPA3 corresponds to stabilizing $\Gamma^A_3 \subset \Gamma^A_2$, where
$$
\Gamma^A_3 = \{ \chi \in \Gamma^A_2 : L(3)(x_p(\chi) - \alpha) = 0 \}.
$$

**SPA4.** While in the formation of SPA3, acquire the desired relative headings, and speed. This corresponds to stabilizing the set $\Gamma^A_4 \subset \Gamma^A_3$, given by
$$
\Gamma^A_4 = \{ \chi \in \Gamma^A_3 : L(b_1 - \beta) = 0 \mod 2\pi, u^i(\chi) = v, \quad w^i_2(\chi) b^2_i + w^i_3(\chi) b^3_i = \frac{v}{\|a\|}, \quad i \in \mathbf{n} \}.
$$

SPA1-4 bear a hierarchical nature where, for $i = 2, 3, 4$, $\Pi^A_1$ is met, i.e. $\chi \in \Gamma^A_1$, only if $\Pi^A_i - 1$ is met. The following control design will reflect this nature in four steps.

**A. Solution of SPA1**

Let
$$
e^i(\chi) = \begin{bmatrix} e^i_1(\chi) \\ e^i_2(\chi) \end{bmatrix} = \begin{bmatrix} \hat{x}^i - u_i(\chi) b^1_i \\ \omega^i - w^i(\chi) \end{bmatrix}, \quad i \in \mathbf{n},
$$
and consider the candidate Lyapunov function $V = \frac{1}{2} \sum_{i=1}^n (e^i)^\top e^i$. The time derivative of $V$ along $\Pi$ (1) is
$$
\dot{V} = \sum_{i=1}^n \left( (\hat{x}^i - u_i(\chi) b^1_i - u_i(\chi) \cdot b^1_i)^\top (\hat{\omega}^i - \hat{w}^i(\chi))^\top \right),
$$
where $\dot{u}_i(\chi) := \frac{\partial w_i(\chi)}{\partial \chi} \hat{\omega}^i + \frac{\partial w_i(\chi)}{\partial \chi} \hat{u}_i$, $i \in \mathbf{n}$. Using $f^i = m_a b^1_i$, $f^i = \bar{m}_a b^1_i$ and $\tau^i = (\hat{\omega}^i)^\top \bar{I}_a \hat{\omega}^i$, $\tau^i = \bar{I}_a (\hat{w}^i(\chi) - k_\tau \omega^i - \hat{w}^i(\chi))$, $i \in \mathbf{n}$, with constants $k_f, k_\tau > 0$, one gets
$$
\dot{V} = -\sum_{i=1}^n k_f \|e^i_1\|^2 + k_\tau \|e^i_2\|^2.
$$

By this, $\Gamma^A_1$ is globally exponentially stable for the closed-loop system (1), (9), provided that all closed-loop solutions are well defined. This can be shown to apply if $u_i(\chi), w^i(\chi), i \in \mathbf{n}$, are chosen to be functions of only $x$, $B_i$ and $u_i(\chi)$ is uniformly bounded on $\chi$. The next result follows.

**Proposition III.1.** Consider system (1), and let $u_i(\chi), w^i(\chi), i \in \mathbf{n}$, be $C^1$ functions of only $x$, $B$, where $u_i(\chi)$ is uniformly bounded on $\chi$. Then, the feedbacks $f^i(\chi), \tau^i(\chi)$ in (9) render the set $\Gamma^A_1$ globally exponentially stable for the closed-loop system (1), (9).

**B. Solution of SPA2**

The objective here is to design $u_i(\chi), w^i(\chi)$, of the previous step, to asymptotically stabilize $\Gamma^A_2$ in (5) relative to $\Gamma^A_1$ for the closed-loop system (1), (9). Note that the closed-loop dynamics restricted to $\Gamma^A_2$ takes the form
$$
\dot{x}^i = u_i(\chi) b^1_i, \quad \dot{B}_i = B_i(w^i(\chi))^\times, \quad i \in \mathbf{n},
$$
(11)
Let
$$
\dot{w}^i = \hat{w}^i + \bar{w}^i + \bar{w}^i, \quad i \in \mathbf{n},
$$
and pick $u^i(\chi), \hat{w}^i(\chi)$, $\hat{w}^i(\chi)$ such that $\Gamma^A_2$ is invariant for
$$
\dot{x}^i = u^i(\chi) b^1_i, \quad \dot{B}_i = B_i(\hat{w}^i(\chi) + \bar{w}^i(\chi))^\times, \quad i \in \mathbf{n}.
$$
(13)

For $i \in \mathbf{n}$, let $w^i(\chi)$ be any $C^1$ function, and
$$
\dot{w}^i_2 = (a \cdot b^2_i) \frac{u^i(\chi)}{r}, \quad \dot{w}^i_3 = (a \cdot b^3_i) \frac{u^i(\chi)}{r},
$$
(14)
and
$$
\dot{w}^i_2 = (a \cdot b^2_i) \frac{\delta \tilde{w}^i_2(\chi)}{r}, \quad \dot{w}^i_3 = (a \cdot b^3_i) \frac{\delta \tilde{w}^i_3(\chi)}{r},
$$
(14)
where $\delta \tilde{w}^i_2(\chi) = \tilde{w}^i_2(\chi) - \alpha_0$.

Let $\tilde{w}^i_2 = \tilde{k} (a \cdot b^2_i + k_\tau) \psi_i(\chi)$, $\tilde{w}^i_3 = \tilde{k} (a \cdot b^3_i + k_\tau) \psi_i(\chi)$, where $\tilde{k}, \tilde{k} > 0$, and
$$
\psi_i(\chi) = \sum_{j \in N(i)} B^T_j (x^i - x^j) \cdot B^T_j a - (\alpha_3 - \alpha_i) \cdot a.
$$
(15)

Consider the $n \times n$ orthonormal matrix
$$
P = \begin{bmatrix}
\sqrt{1} & \sqrt{2} & \cdots & \sqrt{(n-1)(n-2)} & \sqrt{(n-1)(n-3)} & \cdots & \sqrt{n-1} \\
0 & \sqrt{(n-2)(n-3)} & \cdots & \sqrt{(n-1)(n-3)} & \sqrt{n-1} & \cdots & 0 \\
0 & 0 & \cdots & \sqrt{n-1} & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & 0 & \cdots & \sqrt{1} \\
\end{bmatrix},
$$
and let
$$
y = P^{-1}(x_a - \alpha_a), \quad z = P^{-1}(a \cdot b^1_i \cdots a \cdot b^1_i)^\top.
$$
(16)

Using (14), (15), and the fact that $\sum_{j \in N(i)} (x^i - x^j) \cdot a - (\alpha_3 - \alpha_i) \cdot a = L^i(x_a - \alpha_a)$, the dynamic $\hat{y}, \hat{z}$ takes the form
$$
\hat{y} = vz + \tilde{k} P^{-1} U(b_1) P z,
$$
$$
\hat{z} = -\tilde{k} P^{-1} A(z) \left( P z + k(v I_{m_a \times n_a} + kU(b_1)) L p \right),
$$
where $A(z) = \text{diag}(1 - (a \cdot b^1_i)^2, \cdots, 1 - (a \cdot b^1_i)^2)$, and $U(b_1) = \text{diag}(\tilde{u}_1(b_1), \cdots, \tilde{u}_n(b_1))$. Since $L$ is balanced, it follows that $L^T$ is a Laplacian of a digraph which has a globally reachable node as well. From this, and the fact that the first column of $P$ lies in ker $L$, it follows that $P^{-1} L P = \text{diag}(0, M)$, for some matrix $M \in \mathbb{R}^{n-1 \times n-1}$ whose eigenvalues have strictly positive real parts. Now, consider the
partition $y = [\bar{y} \  \tilde{y}^\top]^\top$, where $\bar{y} \in \mathbb{R}, \tilde{y} \in \mathbb{R}^{n-1}$. Substituting this in (17) gives
\[ \dot{\bar{y}} = vz_1 + \hat{k} \bar{D}(b_1)z, \quad \dot{\tilde{y}} = vz_2 + k(vu_{n \times n} + kU(b_1))P[0 \ \tilde{y}^\top M]^\top, \]
with $\bar{D}(\text{SO}(3))^n \rightarrow \mathbb{R} \times \mathbb{R}^n, \tilde{D}(\text{SO}(3))^n \rightarrow \mathbb{R}^{n \times 1}$ where $[\bar{D}^T(b_1) \ \tilde{D}^T(b_1)]^T = P^{-1}U(b_1)P$. Note that, since $P[y \ 0_{1 \times n-1}]^T \in \ker L, \Gamma_2^A$ can be rewritten as $\chi \in \Gamma_1^A : \dot{\bar{y}} = 0, z = 0$, and so, $\Gamma_2^A$ is asymptotically stable for (11) if the origin of the $\bar{y}, z$ sub-system of (18) is asymptotically stable. This sub-system can be viewed as the nominal system
\[ \dot{y} = [z_2 \ldots z_n]^\top, \quad \dot{z} = -kP^{-1}A(z)(Pz+vkP[0 \ \tilde{y}^\top M]^\top), \]
with perturbation $p(\chi) = k[z^\top D^T(b_1) \ k\tilde{y}^\top M^T \tilde{D}^T(b_1)]^\top$. Since $A(z), U(b_1)$ are uniformly bounded, $p(\chi)$ satisfies $||p(\chi)|| \leq c\hat{k}||\tilde{y}^\top z-\bar{y}^\top M||$, for some $c > 0$. Thus, by Lemma 9.1 in [10], $\hat{k}$ could be chosen small enough such that the origin of the $\bar{y}, z$ sub-system of (18) is exponentially stable if that of (19) is exponentially stable. The linearization of (19) around the origin $\dot{y}(t) = 0$ takes the form
\[ \dot{\tilde{y}} = [z_2 \ldots z_n]^\top, \quad \dot{z} = -\tilde{k}z - vk\hat{k}[0 \ \tilde{y}^\top M]^\top. \]
Using the properties of $L$, it can be shown that, for sufficiently small $k$ the matrix
\[
\begin{bmatrix}
0_{n \times n} & vI_{n \times n} \\
-vk\hat{k}L & -k\bar{D}(b_1)
\end{bmatrix}
\]
is asymptotically stable relative to $\Gamma_1^A$ for the closed-loop system (11), (9).

Proposition III.2. Consider the $n$ kinematic particles in (11). For $i \in n$, $\dot{u}_i(b_1)$ any $C^1$ uniformly bounded function, $w_i^x(\chi), \delta_i(\chi)$ any $C^1$ functions of only $x, B$; $\bar{k}, \hat{k}$ could be chosen small enough such that the feedbacks (12), (14), (15) render $\Gamma_2^A$ asymptotically stable for (11). Therefore, the feedbacks (9), with $u_i(\chi), w^x_i(\chi)$ as given previously, render $\Gamma_2^A$ asymptotically stable relative to $\Gamma_1^A$ for the closed-loop system (11), (9).

C. Solution of SPA3

The objective here is to design $\delta_i(\chi)$ such that $\Gamma_3^A$ is asymptotically stable for (13), and such that $\delta_i(\chi) = 0$ on $\Gamma_3^A$. Notice that, from the solution of SPA2, $\dot{u}_i(\chi) = v + \tilde{k}\dot{u}_i(b_1), \ w_i^x(\chi)b_2^i + \bar{w}_i^x(\chi)b_2^i = v + \tilde{k}\hat{u}_i(b_1) + \bar{h}_i(\chi)a, i \in n$, on $\Gamma_2^A$. Thus, on $\Gamma_3^A$, each rigid body follows a circle with radius $r$, and stationary center. Therefore, $\Gamma_3^A$ is invariant for
\[ \dot{x}^i = w_i^x(\chi)b_1^i, \quad \dot{B}_i = B_i^i(w_i^x(\chi))^X, \quad i \in n. \]
Let
\[ \delta_i(\chi) = k \sum_{j \in N(i), j \neq i} (B_j^i(x_p^j - x_p^i))^\top e_j - (\alpha_i - \bar{\alpha_i})^\top b_1^i, \]
where $\bar{k} > 0$. Using this, one gets the following.

Proposition III.3. Consider the dynamics (13) representing the motion of the rigid bodies on $\Gamma_3^A$. For $i \in n$, $\dot{u}_i(b_1), w_i^x(\chi)$ as in Proposition III.2, $\bar{k} > 0$ could be chosen small enough such that there exists $\bar{k}^* > 0$ where, for all $\bar{k} \in (0, \bar{k}^*)$ the feedback (12), (14), (15), (22) renders $\Gamma_3^A$ asymptotically stable for (13). Therefore, the feedbacks (9), with $u_i(\chi), w^x(\chi)$ as given previously, render $\Gamma_3^A$ asymptotically stable relative to $\Gamma_2^A$ for the closed-loop system (1), (9).

The proof of this result, as well as Theorems III.5, IV.1 and Corollary V.1 to follow, is omitted due to space limitations, and will be provided elsewhere.

D. Solution of SPA4

Here, $\dot{u}_i(b_1)$ is designed to stabilize $\Gamma_4^A$ in (7) for (21). From the previous step, on $\Gamma_3^A$ all rigid bodies follow circles with radius $r$, and fixed center. Their motion, therefore, can be completely parameterized by $b_1$. Since on $\Gamma_3^A, u_i(\chi) = v + \tilde{k}\dot{u}_i(b_1), \ w_i^x(\chi)b_2^i + \bar{w}_i^x(\chi)b_2^i = u_i(\chi)a$, then
\[ \dot{b}_i^1 = (1/r)(v + \tilde{k}\dot{u}_i(b_1)), \quad i \in n. \]
Thus, to stabilize $\Gamma_4^A$ for (21) one needs to design $C^1$ feedbacks $\dot{u}_i(b_1)$ that asymptotically stabilize the set $\{b_1 : L(b_1 - \beta) = 0 \mod 2\pi \}$ for (23), and in addition, on that set it must hold that $\dot{u}_i(b_1) = 0, i \in n$. The next result follows directly from Proposition V.4 in [1].

Proposition III.4. For $\bar{k} > 0, i \in n, \dot{u}_i = -\sin\left(\sum_{j \in N(i)}(\epsilon_1, a^x_j - \angle(B_i^j b_1^j, a^x_i) - (\beta_i - \beta_j))\right), \quad (24) \]
where $a^x_j = B_i^j a^x_j$, render the set $\{b_1 : L(b_1 - \beta) = 0 \mod 2\pi \}$ asymptotically stable for (23). Hence, the feedback (12), (14), (15), (22), (24) renders $\Gamma_4^A$ asymptotically stable for (21), and so, the feedbacks (9), with $u_i(\chi), w^x(\chi)$ as given previously, render $\Gamma_4^A$ asymptotically stable relative to $\Gamma_3^A$ for the closed-loop system (1), (9).

E. Solution of CFPR - Case A

The solution of CFPR for fully actuated rigid bodies follows next, from that of SPA1-4, and Proposition 14 in [3].

Theorem III.5. Consider the $n$ rigid bodies in (1), and assume that $G$ is balanced, and that it has a globally reachable node. For $i \in n$, $w_i^x(\chi)$ any $C^1$ function of only $x, B$, there exists $k^*, k^*$ such that for all $k \in (0, k^*), k \in (0, k^*)$, $\bar{k}, \hat{k} > 0$ small enough, the feedbacks (9) with $u_i(\chi) = v + \tilde{k}\dot{u}_i(b_1), \ w_i^x(\chi)$ given in (24); and $w_i^x(\chi), w_i^x(\chi)$ in (15), (14), (22), (23), solve CFPR, rendering the sets $\Gamma_3^A$ in (6) and $\Gamma_4^A$ in (7) asymptotically stable for the closed-loop system. In addition, for initial conditions in a neighbourhood of $\Gamma_3^A$, the rigid bodies converge to stationary formations.

Remark 1. The result in Theorem III.5 encompasses a solution of the circular formations problem for steered kinematic particles. In [11], the problem was addressed for particles of the form (11) with undirected information graph.

IV. CASE B: UNDERACTUATED BODIES - DEGREE ONE

In this section CFPR is solved under the assumption that (1) has one degree of underactuation where $f_3^j = 0$. 3237
In this case CFPR is broken down into six sub-problems. 

**SPB1.** Design feedbacks $\tau^i(\chi), i \in n$, stabilizing the set 
\[ \Gamma^i_1 = \{ \chi \in \mathcal{X} : \omega^i = w^i(\chi), \ i \in n \}, \]
where $w^i(\chi), i \in n$, are $C^1$ functions to be designed later.

**SPB2.** For the motion on $\Gamma^i_1$, make the orientations of the rigid bodies be such that $f^j_i = 0$, $i \in n$, for the feedbacks in (9). This can be achieved by stabilizing the set $\Gamma^i_2 \subset \Gamma^i_1$, where 
\[ \Gamma^i_2 = \{ \chi \in \Gamma^i_1 : f^j_i(\chi) = 0, \ i \in n \}, \]
with $k_f > 0$, and $w^i(\chi)$ as in Theorem III.5.

**SPB3.** Design $f^j_1(\chi), f^j_2(\chi)$ to obtain desired linear velocities on $\Gamma^i_2$, i.e. stabilize the set 
\[ \Gamma^i_3 = \{ \chi \in \Gamma^i_2 : \dot{x}^i = u_i(\chi)b^i_1, \ i \in n \}, \]
where $u^i(\chi), i \in n$, is defined in Theorem III.5. Note that the dynamics on $\Gamma^i_3$ are that of steered kinematic particles, where $\Gamma^i_3 \subset \Gamma^i_2$.

**SPB4.** This sub-problem is similar to SPB4.2. The objective is to stabilize the set $\Gamma^i_4$ on $\Gamma^i_3$, where $\Gamma^i_4 = \Gamma^i_3 \cap \Gamma^i_2$.

**SPB5.** This sub-problem is similar to SPB3.3. The objective is to stabilize the set $\Gamma^i_5$ on $\Gamma^i_4$, where $\Gamma^i_5 = \Gamma^i_4 \cap \Gamma^i_3$.

**SPB6.** This sub-problem is similar to SPB4.4. The objective is to stabilize the set $\Gamma^i_6$ on $\Gamma^i_5$, where $\Gamma^i_6 = \Gamma^i_5 \cap \Gamma^i_3$.

**A. Solution of SPB1**

Recall $e^i(\chi) = \omega^i - w^i(\chi)$ in (8), and consider the candidate Lyapunov function $V = \frac{1}{2} \sum_{i=1}^{n} (e^i)^T e^i$, and 
\[ \tau^i = (\omega^i)^T \omega^i + \sum_{i=1}^{n} \left[ w^i(\chi) - k_r(\omega^i - w^i(\chi)) \right], \ i \in n, \]
with $k_r > 0$, in (9). Using this, $V$ takes the form
\[ V = -2k_r V, \]
and so, $\Gamma^i_1$ is asymptotically stable.

**B. Solution of SPB2**

Since on $\Gamma^i_1$, $\omega^i = w^i(\chi)$, then, on that set, $b^i_1 = w^i_3(b^i_2 - w^i_3(b^i_3))$. It thus follows, from (27), that 
\[ f^j(\chi) = 0 \implies w^i_3(u_i(\chi)b^i_1 + k_f(b^i_3)^T x^i) = 0. \]
Consider the feedbacks 
\[ f^j_i = m_i(b^i_j)^T \left[ u_i(\chi)b^i_1 + u_i(\chi)b^i_1 - k_f(x^i - u_i(\chi)b^i_1) \right], \]
where $u_i(\chi), w^i_2(\chi), w^i_3(\chi)$ are defined as in Theorem III.5, with $\delta_i, \psi_i$ replaced by $k_f(b^i_3)/k_f$, $\varphi_i$ where $\varphi \in \text{Sat}(\mathbb{R})$. Note that $w^i_2(\chi)$ can be rewritten as $\xi^i = [a \cdot b^i_2 \cdot a \cdot b^i_3]^T = [\theta_1 \theta_2]$, 
\[ \xi^i = \left[ \frac{1}{k} \left( u_i(\chi) + k_\varphi(L^i_1(x_a - \alpha_a)) b^i_1 \right) \right]. \]
By this, (31) is satisfied if $\theta_1 - \theta_2 = \cos^{-1} \frac{-k_f(b^i_3)^T x^i}{u_i(\chi)b^i_1} [\theta_1, \theta_2]$, provided that this angle is well defined. Consider the function 
\[ \Theta = \frac{1}{2} \sum_{i=1}^{n} (\theta_1 - \theta_2 - \bar{\theta}_i)^2, \]
where $\bar{\theta}_i = \cos^{-1} \frac{k_f(b^i_3)^T x^i}{u_i(\chi)b^i_1}$. Since $b^i_2, b^i_3$ do not appear in $\xi^i$, it follows that $\bar{\theta}_i$ is not a function of $w^i_1$. Taking the time derivative of $\theta_1, \bar{\theta}_i$ along the dynamics on $\Gamma^i_1$ gives 
\[ \dot{\bar{\theta}}_i = -w^i_2 b^i_3 + cos \bar{\theta}_i u_i(\chi)(\xi^i)^T [\xi^i] \sqrt{(u_i(\chi)(\xi^i)^T [\xi^i])^2 - (k_f(b^i_3)^T x^i)^2}. \]
It is easy to show that $w^i_3$ does not appear in $u_i(\chi)(\xi^i)^T [\xi^i]$. Let $\bar{\xi} = \sqrt{(u_i(\chi)(\xi^i)^T [\xi^i])^2 - (k_f(b^i_3)^T x^i)^2}$. By setting 
\[ w^i_2 = \cos \bar{\theta}_i u_i(\chi)(\xi^i)^T [\xi^i] - w^i_2 k_f(b^i_3)^T x^i + \kappa_i(\chi) - k_f(b^i_3)^T x^i \]
and 
\[ \kappa_i(\chi) = -k_f(b^i_3)^T x^i \]
where $k_{w_i} > 0$, the time derivative of $\Theta$ in (33) along the system dynamics on $\Gamma^i_1$ becomes 
\[ \dot{\Theta} = -k_{w_i} \sum_{i=1}^{n} (\theta_1 - \theta_2 - \bar{\theta}_i)(\theta_1 - \theta_2 - \bar{\theta}_i). \]
By this, and the fact that $\bar{\Theta}, \bar{\xi}$ are uniformly bounded, it follows that, for initial conditions in $T = \{ \chi \in \Gamma^i_1 : |\theta_1(\chi) - \theta_2(\chi) - \theta_1(\chi)| \leq c_0 < \pi, i \in n \}$, $w^i_2(x(t))u_i(x(t)) + k_f(b^i_3)^T x^i(t) \rightarrow 0$ exponentially, and so, $\Gamma^i_2$ is asymptotically stable relative to $\Gamma^i_1$ if $w^i_1(\chi)$ is well defined. To this end, let $\bar{k}, \hat{k} > 0$ be chosen small enough such that $u_i(\chi), |\xi^i| > 0, i \in n$, for all $\chi \in X$. Now, consider 
\[ V_i = \frac{1}{2} (a \cdot b_i)^2, \ i \in n. \]
The time derivative of $V_i$ along the dynamics on $\Gamma^i_1$ is 
\[ \dot{V}_i = (a \cdot b_i)(a \cdot (w^i_2(\chi)b^i_2 - w^i_2(\chi)b^i_3)) = -(\bar{k}(a \cdot b_i) + k_\varphi(L^i_1(x_a - \alpha_a)))) - (a \cdot b^i_2)^2] \]
Let $k < \frac{1}{2(k_f + k)}$. Using this, the fact that $|\varphi(\cdot)| < 1$, and the previous $V_i$, it follows that the set $\mathcal{V} = \{ \chi \in \Gamma^i_1 \cap \Gamma^i_2 : \chi(\chi) < \frac{1}{2(k_f + k)} \}$ is positively invariant. On this set, $|\xi|^i \geq c_0 > 0$, $i \in n$. Also, by the previous choice of $\bar{k}, |\xi_i| \geq \frac{\sqrt{k_f + k}}{r} > 0$, $i \in n$, for all $\chi \in X$. Next, consider the diffeomorphism 
\[ e^i = x_i - u_i(\chi)b^i_1 \mapsto e^i, \ b^i = B_i e^i \]
and the functions 
\[ W_i = \frac{1}{2}(e^i)^T e^i, \ i \in n. \]
The time derivative of $W_i$ along the dynamics on $\Gamma^i_1$ is 
\[ \dot{W}_i = -2k_f W_i + e^i(e^i)^T u_i(\chi) + k_f(b^i_3)^T x^i \]
\[ + u_i(\chi)^T \sqrt{2W_i} \left( \cos \theta_i - \theta_{\xi_i} + k_f(b^i_3)^T x^i \right) \]
where $u_i = v + \bar{k}, \ \bar{\xi} = 1,$ and $\xi_i = \sqrt{(v + \bar{k})^2 + \bar{k} + (\bar{k}(v + \bar{k}) + \bar{k})^2}$. Consider the sub-level set $W = \{ \chi \in \mathcal{X} : W_i < W_0, i \in n \}$, and let $k_f < \frac{1}{(v + \bar{k})^2 + \bar{k} + (\bar{k}(v + \bar{k}) + \bar{k})^2}$. Using this, (35), and (39), it follows that, for sufficiently small $c_0$, $\mathcal{V} \cap \mathcal{W} \cap \mathcal{T}$ is positively invariant for the closed-loop system. $w^i_1(\chi), i \in n$, in (34) is well defined for initial conditions on $\mathcal{V} \cap \mathcal{W} \cap \mathcal{T}$, which is a neighbourhood of $\Gamma^i_2$ on $\Gamma^i_1$. 

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C. Solution of SPB3

Consider the feedbacks (32), the diffeomorphism (37), and the functions (38). The time derivative of $W_i$ along the system dynamics on $\Gamma_B^i$ takes the form $W_i = -2k_f W_i$. This implies that $\Gamma_B^i$ is asymptotically stable relative to $\Gamma_B^i$.

D. Solution of CFPR - Case B

From the solutions of SPB1-3, the system dynamics on $\Gamma_B^3$ takes the form (11). The solutions of SPB4-6 follow as that of SPB3.

Theorem IV.1. Under the conditions of Theorem III.5, there exists $k^*, k^*$ such that for all $k \in (0, k^*)$, $k \in (0, k^*)$, $k, k, k_f > 0$ such that, the feedbacks (29), (32), $w_i^1(\chi)$ as in (34), and $u_i(\chi)$, $w_2(\chi)$, $w_3(\chi)$ as in Theorem III.5 with $\delta_1(\chi), \psi(\chi)$ replaced by $k \varphi(\delta_1(\chi)/k), \varphi(\psi(\chi))$, solve CFPR for underactuated rigid bodies with $f_i^j = 0$, rendering the sets $\Gamma_B^i$ and $\Gamma_B^j$ asymptotically stable for the closed-loop system. In addition, for initial conditions in a neighbourhood of $\Gamma_B^i$ the rigid bodies converge to stationary formations.

V. CASE C: UNDERACTUATED RIGID BODIES - DEGREE TWO

In this section system (1) is assumed to have two degrees of underactuation: $f_1^j = f_3^j = 0$, $i \in n$. Control design to solve CFPR without specification (iii) is achieved by breaking down the problem into five sub-problems of stabilizing five nested sets $\Gamma_C^1 \supset \cdots \supset \Gamma_C^5$. These sub-problems are similar to SPB1-5, of the previous section, where $\Gamma_j^i = \Gamma_j^j$, $j = 1, \cdots, 5$. The development is omitted here due to space limitations, and will be provided elsewhere.

Corollary V.1. Under the conditions of Theorem III.5, there exists $k^*, k^*$ such that, for all $k \in (0, k^*)$, $k \in (0, k^*)$, $k, k, k_f > 0$ small enough, the feedbacks (29), (32), $u_i(\chi) = v$, and $w^i(\chi)$ as in Theorem IV.1, solve CFPR without specification (iii) for underactuated (1) where $f_i^j = f_3^j = 0$, $i \in n$, rendering the set $\Gamma_C^5 = \Gamma_B^5$ asymptotically stable for the closed-loop system.

VI. SIMULATIONS

This section presents simulation results using the feedback in Theorem III.5 for 5 rigid bodies with a cyclic Laplacian where $L_1 = [1 - 1 0 0 0]$.

Formation 1. The formation in Figure 2 is given by

\[
   r = 1, \quad a = \begin{bmatrix} 0 & \sin 30^\circ & \cos 30^\circ \end{bmatrix}^\top, \quad \theta_1 = \theta_2 = \theta_3 = \theta_4 = 0, \quad p_1 = T[5 \cos 72^\circ 5 \sin 72^\circ 0]^\top, \quad p_2 = T[5 \cos 288^\circ 5 \sin 288^\circ 0]^\top, \quad \text{and} \quad p_3 = \begin{bmatrix} 5 \cos 216^\circ 5 \sin 216^\circ 0 \end{bmatrix}^\top.
\]

This gives coplanar circular paths centered at the edges of a pentagon.

Formation 2. The formation in Figure 3 in given by $r = 2$, $a = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^\top$, $p_1 = p_2 = p_3 = p_4 = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}^\top$, and $\theta_1 = \theta_2 = \theta_3 = \theta_4 = 2\pi$. In this formation, the circular paths are coplanar and concentric.

REFERENCES


