Left inversion of nonlinear time delay system

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Abstract—This paper investigates the left invertibility for nonlinear time delay system with internal dynamics. Under the assumption imposed on the internal dynamics, it has been shown that the unknown inputs can be estimated. Causal and non causal estimation of the unknown inputs are respectively discussed, and the high-order sliding mode observer is used to estimate the observable states.

I. INTRODUCTION

In many applications, recovering of unknown inputs from the outputs is crucial. This may occur for example in data secure transmission, where the unknown input is the message or in fault detection and isolation where the fault is the unknown input. It is the reason why this problem was studied since at least forty five years ago in linear control theory [16], [18] and thirty five year ago in nonlinear control theory [6], [17]. Most of those works are in the context of nonlinear systems without time delays. Considering nonlinear systems with time delays, an important tool based on non-commutative ring was introduced in [19]. This tool of non commutative ring [9] is used to model nonlinear time delayed system in algebraic framework, and many results are already obtained. The notions of Lie derivatives and relative degree are defined, and the differences between the causal and non-causal invertibility are clarified in [22]. The canonical form of invertibility is also given in [21], and in [20] a method for estimating of the unknown inputs is proposed. However, the algorithm for left invertibility proposed in [21] is only for the system without internal dynamics. For system with or without time delay, the main difficulty when the internal dynamics (or inverse dynamics [3]) occurs is to estimate the state of such dynamics. One interesting solution in order to overcome such a difficulty is to allow the derivative of the unknown inputs [15] with a geometrical approach and with an algebraical one. If however the input derivatives are not possible, then it is necessary to compute and analyze the internal dynamics.

For nonlinear systems without delay, if the vector fields associated to the inputs verify some involutivity properties, then the internal dynamic does not depend on the unknown input. However, this rule is not valid for nonlinear systems with time delay. In order to analyze the internal dynamics and estimate its state, one solution is to rewrite this internal dynamics such that its dynamic becomes independent of the unknown input [1]. Thus in the paper we will adapt this new way to determine the internal dynamics for the nonlinear systems with time delay. This method is based on the finite time convergence by using the existing observer for time delay system in the literature [5], [7], [8], [10], [2] and the high order sliding mode proposed in [12] is applied in this paper.

The paper is organized as follows: In the next section the algebraic framework and some notations are recalled. In section III, we firstly propose a canonical form based on the previously proposed algorithm in [21]. After that the inverse dynamics is computed and sufficient condition for causal and non causal left invertibility are discussed. This section ends with an observer scheme dedicated to the proposed canonical form. In section IV two examples and numerical results highlight the feasibility and the efficiency of the proposed approach.

II. ALGEBRAIC FRAMEWORK AND NOTATIONS

Consider the following class of nonlinear time delay system:

\[
\begin{align*}
\dot{x} &= f(x(t-j\tau)) + \sum_{i=1}^{m} g_i(x(t-j\tau))u_j(t) \\
x &= \psi(t), u(t) = \varphi(t) \\
t &\in [-\tau, 0]
\end{align*}
\]

with \(x \in W \subset \mathbb{R}^n\) is the state vector of the system, \(u(t) \in \mathbb{R}^m\) is the vector of its unknown input and \(y \in \mathbb{R}^p\) is the output. \(\tau\) represents the basic commensurate time delay.

In [19], an algebraic framework is developed, defining the field \(K\) of a finite number of the variables from:

\[
\{x_j(t-i\tau), j \in [1, n], i \in [0, s] \subset N\}
\]

such that \(E\) represents the vector space over \(K : E = \text{span}_K \{d\xi : \xi \in K\}\) and \(\tau\) represents the basic commensurate time delay. Denote the operator \(\delta\) as a backward shift operator, which means \(\delta^i\xi(t) = \xi(t-i\tau)\) and \(\delta^i(a(t)d\xi(t)) = a(t-i\tau)d\xi(t-i\tau)\). With this operator, we can then define the following set of polynomials \(K[\delta]\):

\[
a(\delta) = a_0(t) + a_1(t)\delta + \cdots + a_r(t)\delta^r, a_i(t) \in K
\]

The addition for the entries in \(K[\delta]\) is defined as usual, but its multiplication is given by the following criteria:

\[
a(\delta)b(\delta) = \sum_{k=0}^{r_a+r_b} \sum_{i+j=k} a_i(t)b_j(t-i\tau)\delta^k
\]
With the standard differential operator $d$, denote by $\mathcal{M}$ the left module over $\mathcal{K}(\delta)$:

$$\mathcal{M} = \text{span}_{\mathcal{K}(\delta)} \{d\xi, \xi \in \mathcal{K}\}$$  \hspace{1cm} (5)

Hence, $\mathcal{K}(\delta)$ is a non-commutative ring satisfying the associative law, and it has been proved in [9] and [19] that it is a left Ore ring, which enables us to define the rank conception.

Then, for any matrix with entries belonging to $\mathcal{K}(\delta)$, we can introduce the following definition.

Definition 1: (Unimodular matrix) A matrix $A \in \mathcal{K}^{n \times n}(\delta)$ is said to be unimodular over $\mathcal{K}(\delta)$ if it has a left inversion $A^{-1} \in \mathcal{K}^{n \times n}(\delta)$, such that $A^{-1}A = I_{n \times n}$.

Based on the above algebraic framework a nonlinear time delay system can be represented in a compact algebraic form as follows:

$$\begin{align*}
\dot{x} &= f(x, \delta) + \sum_{i=1}^{m} G_i(x, \delta)u_i(t) \\
y &= h(x, \delta) \\
x &= \psi(t), u(t) = \varphi(t) \\
t &\in \left[ -\tau, 0 \right]
\end{align*}$$  \hspace{1cm} (6)

where the notation $f(x, \delta)$ means $f(x, \delta) = f(x, x(t-\tau), \ldots, x(t-\tau^r))$ and the same is considered for $G(x, \delta)$ and $h(x, \delta)$.

Based on the above algebraic framework the relative degree can be defined in the following way.

Definition 3: (Relative degree) System (6) has the relative degree $(\nu_1, \ldots, \nu_p)$ in an open set $W \subseteq \mathbb{R}^n$ if the following conditions are satisfied for $1 \leq i \leq p$:

1) for all $x \in W$, $L_{G_i} \dot{h}_i = 0$ for all $1 \leq j \leq m$ and $0 \leq r < \nu_i - 1$;

2) there exists $x \in W$ such that $\exists j \in \{1, \ldots, m\}$, $L_{G_j} L_{\nu_i-1}^{-1} h_i \neq 0$.

If the first condition is satisfied for all $r \geq 0$ and some $i \in \{1, \ldots, p\}$, we set $\nu_i = \infty$.

Moreover, for system (6), one can also define observability indices [11] over non-commutative rings. For $1 \leq k \leq n$, let $\mathcal{F}_k$ be the following left module over $\mathcal{K}(\delta)$:

$$\mathcal{F}_k := \text{span}_{\mathcal{K}(\delta)} \{dh, dL_f h, \ldots, dL_f^{k-1} h\}.$$  

It was shown that the filtration of $\mathcal{K}(\delta)$-module satisfies $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \cdots \subset \mathcal{F}_n$, then define $d_1 = \text{rank}_{\mathcal{K}(\delta)} \mathcal{F}_1$, and $d_k = \text{rank}_{\mathcal{K}(\delta)} \mathcal{F}_k - \text{rank}_{\mathcal{K}(\delta)} \mathcal{F}_{k-1}$ for $2 \leq k \leq n$.

Let $k_i = \text{card} \{d_k \geq i, 1 \leq k \leq n\}$, then $(k_1, \ldots, k_p)$ are the observability indices. Reorder, if necessary, the output components of (6) so that

$$\text{rank}_{\mathcal{K}(\delta)} \{\partial_{x_1} h_1, \ldots, \partial_{x_i} L_{\nu_i}^{-1} h_1, \ldots, \partial_{x_j} L_f h, \ldots, \partial_{x_j} L_f^{p} h_p\} = k_1 + \cdots + k_p.$$  

Definition 4: (Change of coordinates) If there exists $\phi^{-1} \in \mathcal{K}^{n \times 1}$ and some constants $c_1, \ldots, c_n \in \mathcal{N}$ such that $\mathbf{diag}\{\delta^r\} x = \phi^{-1}(\delta, z)$ then the change of coordinates $z = \phi(\delta, x) \in \mathcal{K}^{n \times 1}$ is a causal change of coordinates over $\mathcal{K}^{n \times 1}$. The change of coordinate is bicausal over $\mathcal{K}$ if $\max = \{c_1\} = 0$, i.e $x = \phi^{-1}(\delta, z)$.

Based on the above definition, the following result has been reported in [22], [20].

Theorem 1: For $1 \leq i \leq p$, denote by $k_i$ the observability indices and $\nu_i$ the relative degree index for $y_i$ of (6), and note $\rho_i = \min(k_i, \nu_i)$. Then there exists a causal change of coordinates $\phi(\delta, x) \in \mathcal{K}^{n \times 1}$, such that (6) can be transformed into the following form:

$$\begin{align*}
\dot{\xi}_i &= A_i \xi_i + B_i \mu_i \\
\dot{z} &= \alpha(\xi_i, \zeta_i, \delta) + \beta(\xi_i, \zeta_i, \delta) u \\
y &= C_i \xi_i
\end{align*}$$  \hspace{1cm} (10)

where $A_i$ and $B_i$ are in the Brunovsky form. $\xi_i = \left[ h_i, L_f h_i, \ldots, L_f^{\rho_i} h_i \right]^T \in \mathcal{K}^{n \times 1}$, such that $L_f h_i$ is the Lie derivative of the output $h_i$ along the vector field $f$, and

$$\mu_i = L_{G_i} L_{\nu_i}^{-1} h_i = \sum_{j=1}^{m} L_{G_j} L_{\nu_i}^{-1} h_i (x, \delta) u_j \in \mathcal{K}$$

$\alpha \in \mathcal{K}^{l \times 1}$, $\beta \in \mathcal{K}^{l \times 1}$ where $l = n - \sum_{j=1}^{p} \rho_i$ and the change of coordinates is given as:

$$\phi^T(x, \delta) = (\phi_1^T(x, \delta), \phi_2^T(x, \delta)), \quad \phi_1^T(x, \delta) = (h_1, L_f h_1, \ldots, L_f^{\rho_1} h_1, \ldots, h_p, L_f h_p, \ldots, L_f^{\rho_p} h_p)$$

470
and \( \phi_2(x, \delta) \) is an \((n-l)\) dimensional complementary change of coordinates of \( \phi_1(x, \delta) \).

Consider now the last dynamics of \( \dot{\xi}_i \) and the one of \( \dot{\zeta} \) in (10), one can obtain the following dynamics:

\[
\begin{align*}
\mathcal{H}(\dot{\xi}) &= \Psi(\dot{\zeta}, \xi, \delta) + \Gamma(\dot{\zeta}, \xi, \delta)u \\
\dot{\zeta} &= \tilde{f}(\xi, \xi, \delta, u)
\end{align*}
\]

where

\[
\begin{align*}
\Gamma(\dot{\zeta}, \xi, \delta) &= \begin{pmatrix}
L_{G_1}L_{n-1}^{1-h_1} & \cdots & L_{G_m}L_{n-1}^{1-h_1} \\
L_{G_2}L_{n-1}^{1-h_p} & \cdots & L_{G_m}L_{n-1}^{1-h_p}
\end{pmatrix}\bigg|_{x=\Phi^{-1}(\xi, \xi, \delta)} \\
\Psi &= \begin{pmatrix}
L_{f_1}^{(p_1)} \\
\vdots \\
L_{f_p}^{(p_p)}
\end{pmatrix}, \quad \mathcal{H} = \begin{pmatrix}
\xi_1^{(p_1)} \\
\vdots \\
\xi_{p_p}^{(p_p)}
\end{pmatrix}
\end{align*}
\]

Remark 1: Passing from (10) to (11), the substitution of \( x = \Phi^{-1}(\xi, \xi, \delta) \) must be done. Indeed if the change of coordinates is not bicausal, this may generate non causal terms in the equation (11). Let us consider the following system:

\[
\begin{align*}
\dot{x}_1 &= \delta x_2 \\
\dot{x}_2 &= a_1 x_1 + a_2 x_2 + a_3 x_1 + x_1 u \\
\dot{x}_3 &= b_1 x_1 + b_2 x_2 + b_3 x_3 \\
y_1 &= x_1, y_2 &= x_3
\end{align*}
\]

choosing the change of coordinates as: \( z_1 = x_1, \ z_2 = L_f h_1 = \delta x_2, \) and \( z_3 = x_3 \) then we obtain:

\[
\begin{align*}
\dot{z}_1 &= z_2 \\
\dot{z}_2 &= \delta (a_1 z_1 + a_2 \delta^{-1} z_2 + a_3 z_3 + z_1 u) \\
\dot{z}_3 &= b_1 z_1 + b_2 \delta^{-1} z_2 + b_3 z_3 \\
y_1 &= z_1, y_2 &= z_3
\end{align*}
\]

and the obtained system is not causal.

Noticing that the system (10) contains an internal dynamics \( \dot{\zeta} \). For the trivial case where \( \dim \zeta = 0 \), i.e. \( \text{rank}_{K[\delta]} \phi_1(x, \delta) = n \), if \( \text{rank}_{K[\delta]} \Gamma(\xi, \delta) = m \), then there always exists a matrix \( Q \in \mathbb{K}^{m \times p(\delta)} \) such that \( QA = \Gamma \in \mathbb{K}^{m \times m(\delta)} \) with \( \text{rank}_{K[\delta]} \Gamma = m \). In this case, according to (11), one can obtain the estimation of \( u \) as follows:

\[
\dot{u} = \tilde{\gamma}^{-1} \mathcal{Q}(\mathcal{H}(\dot{\xi}), \delta) - \Psi(\dot{\zeta}, \delta).
\]

where \( \tilde{\gamma} \) represents the estimation of \( \gamma \), which in fact can be estimated by using many existing methods, like sliding mode observer, algebraic observer and so on. Moreover, if the matrix \( \tilde{\gamma} \) is unimodular over \( K[\delta] \), the above estimation of \( u \) is obviously causal.

Considering the non-trivial case, where \( \text{rank}_{K[\delta]} \phi_1(x, \delta) < n \) and \( \text{rank}_{K[\delta]} \Gamma(\dot{\zeta}, \xi, \delta) < m \), the internal dynamics \( \dot{\zeta} \) in (10) is not vanished, thus the invertibility of the unknown inputs depends on the internal dynamics. The following section is devoted to treating this non-trivial case.

III. LEFT INVERTIBILITY WITH INTERNAL DYNAMICS

This section will firstly discuss how to iteratively obtain a new canonical form for the non-trivial case, and secondly study the left invertibility of the unknown inputs with internal dynamics in the deduced canonical form. Finally a high order sliding mode observer is designed to estimate the state variables and their derivatives in order to estimate the unknown inputs.

A. Canonical form

Considering system (6), if \( \text{rank}_{K[\delta]} \phi_1(x, \delta) < n \) and \( \text{rank}_{K[\delta]} \Gamma(\xi, \delta) < m \), an algorithm was developed in [22] to generate \( l_\gamma \) new outputs of the systems, which are combinations of \( y_i \), its derivatives and their backward shifts. Without loss of generality, suppose \( \sum_{i=1}^{\gamma} l_i = j \), thus \( \{dh_1, \ldots, dL_{f_1}^{p_1-1} h_1, \ldots, dh_p, \ldots, dL_{f_p}^{p_p-1} h_p\} \) are \( j \) linearly independent vectors over \( K[\delta] \). Then note:

\[
\Phi = \{dh_1, \ldots, dL_{f_1}^{\rho_1-1} h_1, \ldots, dh_p, \ldots, dL_{f_p}^{\rho_p-1} h_p\}
\]

and

\[
\mathcal{L} = \text{span}_{\mathbb{R}[\delta]} \{h_1, \cdots, L_{f_1}^{p_1-1} h_1, \cdots, h_p, \cdots, L_{f_p}^{p_p-1} h_p\}
\]

where \( \mathbb{R}[\delta] \) is the commutative ring of polynomials in \( \delta \) with coefficients belonging to the field \( \mathbb{R} \), and let \( \mathcal{L}(\delta) \) be the set of polynomials in \( \delta \) with coefficients over \( \mathcal{L} \). The module spanned by element of \( \Phi \) over \( \mathcal{L}(\delta) \) is defined as follows:

\[
\Omega = \text{span}_{\mathcal{L}(\delta)} \{\xi, \xi, \Phi\}
\]

Define

\[
\mathcal{G} = \text{span}_{\mathbb{R}[\delta]} \{G_1, \ldots, G_m\}
\]

where \( G_i \) is given in (6), and its left annihilator:

\[
\mathcal{G}^\perp = \text{span}_{\mathcal{L}(\delta)} \{\omega \in \mathcal{M} \mid \omega \beta = 0, \forall \beta \in \mathcal{G}\}
\]

where \( \mathcal{M} \) is defined in (5).

Then the new outputs can be virtually obtained if the sufficient condition \( \text{rank}_{K[\delta]} (\mathcal{H}_\gamma) = l_\gamma \) is satisfied, where

\[
\mathcal{H}_\gamma = \text{span}_{\mathbb{R}[\delta]} \{\omega \in \mathcal{G}^\perp \cap \Omega \mid \omega f \notin \mathcal{L}\}
\]

More explicitly, the new generated outputs are noted as: \( \bar{y}_i = \omega_i f \mod \mathcal{L} \), for \( 1 \leq i \leq l - \gamma \) and \( \omega_i \in \mathcal{H}_\gamma \).

The above algorithm will be iterated until

\[
\text{rank}_{K[\delta]} \phi_1(x, \delta) = n \quad \text{or} \quad \text{rank}_{K[\delta]} \Gamma(\zeta, \delta) = m.
\]

For the final obtained canonical form, if

\[
\text{rank}_{K[\delta]} \phi_1(x, \delta) = n \quad \text{and} \quad \text{rank}_{K[\delta]} \Gamma(\zeta, \delta) = m
\]

then the problem of left invertibility for the unknown inputs cannot be solved. For the second case where \( \text{rank}_{K[\delta]} \phi_1(x, \delta) < n \) and \( \text{rank}_{K[\delta]} \Gamma(\zeta, \delta) = m \), after \( l_\gamma \) iterations, the obtained canonical form can be written as follows:

\[
\begin{align*}
\dot{z}_{i,j} &= z_{i,j+1}, \quad \text{for} \ 1 \leq i \leq p + l_\gamma, \text{ and } 1 \leq j \leq \theta_i - 1 \\
\dot{z}_{i,\theta_i} &= b_i(z, \eta, \delta) + \sum_{j=1}^{m} a_{i,j}(z, \eta, \delta)u_j, \quad \text{for} \ 1 \leq i \leq p + l_\gamma \\
\dot{\eta} &= \tilde{f}_i(z, \eta, \delta, u)
\end{align*}
\]

with \( a_{i,j} = L_{g_j}L_{f_j}^{\theta_i-1} h_i \) and \( b_i(z, \delta, \eta) = L_{f_i}^{\theta_i} h_i \), where \( l_\gamma \) is the number of new outputs, \( \theta_i \) is the minimum value
of the observability index and relative degree for the $i$th output. Then the second and the third equation in (19) can be rewritten in the following compact way:

$$\begin{cases}
\mathcal{H}_\gamma(z) = \Psi_\gamma(z, \eta, \delta) + \Gamma_\gamma(z, \eta, \delta) u \\
\dot{\eta} = \bar{f}_\gamma(\eta, z, u, \delta)
\end{cases} \tag{20}$$

where

$$\Gamma_\gamma(z, \eta, \delta) = \begin{pmatrix}
L_{G_1}^\theta \gamma_{-1} h_1 \\
L_{G_1}^\theta \gamma_{+1} h_p \\
L_{G_1}^\theta \gamma_{+2} h_p+1 \\
\vdots \\
L_{G_m}^\theta \gamma_{-1} h_1 \\
L_{G_1}^\theta \gamma_{+1} h_p \\
L_{G_1}^\theta \gamma_{+2} h_p+1 \\
L_{G_m}^\theta \gamma_{+2} h_p+2
\end{pmatrix}_{|x = \phi^{-1}(z, \delta)}$$

$$\mathcal{H}_\gamma(z) = \begin{pmatrix}
\eta_p \\
\eta_p+1 \\
\eta_p+2 \\
\vdots \\
\eta_p+n
\end{pmatrix}_{|x = \phi^{-1}(z, \delta)}$$

$$\Psi_\gamma(z, \eta, \delta) = \begin{pmatrix}
\bar{L}_f^\gamma \eta_{p+1} h_p+1 \\
\bar{L}_f^\gamma \eta_{p+2} h_p+2 \\
\bar{L}_f^\gamma \eta_{p+3} h_p+3 \\
\vdots \\
\bar{L}_f^\gamma \eta_{p+n} h_p+n
\end{pmatrix}_{|x = \phi^{-1}(z, \delta)}$$

\section{B. Left inversion}

For the iteratively obtained canonical form (19) (or the compact form (20)), if $\text{rank}_{\mathcal{K}(\delta)} \Gamma_\gamma \in m$, then there always exists a matrix $Q_\gamma \in \mathcal{K}(\delta)^{m \times (p+1)}$ such that $Q_\gamma \Gamma_\gamma = \bar{\Gamma}_\gamma \in \mathcal{K}^m \times m(\delta)$ with $\text{rank}_{\mathcal{K}(\delta)} \bar{\Gamma}_\gamma = m$. Then we can obtain:

$$u = \bar{\Gamma}_\gamma^{-1} Q_\gamma (\mathcal{H}_\gamma(z) - \Psi_\gamma(z, \eta, \delta)) \tag{21}$$

Thus, the dynamics of $\eta$ in (20) can be rewritten in the following form:

$$\dot{\eta} = \bar{f}_\gamma(\eta, z, \delta, \hat{z}_{p+t+1}, \hat{z}_{p+t+2}, \ldots) \tag{22}$$

for which we can propose the following estimator:

$$\hat{\eta} = \bar{f}_\gamma(\hat{\eta}, \hat{z}, \delta, \hat{z}_{p+t+1}, \hat{z}_{p+t+2}, \ldots) \tag{23}$$

where $\hat{z}$ represents the estimation of $z$.

\textbf{Assumption 1:} It is assumed that there exists a Lyapunov function $V(e)$ with $e = \eta - \hat{\eta}$ such that for all $\delta$ and $z$:

$$\dot{V} = \frac{\partial V}{\partial e} \left[ \bar{f}_\gamma(\eta, z, \delta) - \bar{f}_\gamma(\hat{\eta}, \hat{z}, \delta) \right] \leq 0 \tag{24}$$

Then we are able to state the following result.

\textbf{Theorem 2:} Under Assumption 1, if $\bar{\Gamma}_\gamma \in \mathcal{K}^m \times m(\delta)$ is unimodular over $\mathcal{K}(\delta)$, then the unknown input $u$ of the system (6) can be causally estimated.

For system (6), since we can use the existing method to estimate $z$ and its derivative in a finite time $T$, then the defined dynamics (23) can asymptotically estimate $\eta$. This is due to the existence of a Lyapunov function $V(e)$ such that $\dot{V}(e) < 0$, imposed by Assumption 1.

After the convergence of $\dot{\eta}$ to $\eta$, the unknown input $u$ of (6) can then be calculated via (28). Moreover, if $\bar{\Gamma}_\gamma \in \mathcal{K}^m \times m(\delta)$ is unimodular over $\mathcal{K}(\delta)$, i.e., $\bar{\Gamma}_\gamma^{-1} \in \mathcal{K}^m \times m(\delta)$, since $Q_\gamma \in \mathcal{K}(\delta)^{m \times (p+1)}$, $\mathcal{H}_\gamma(z, \eta, \delta) \in \mathcal{K}(\delta)^{p+1 \times 1}$ and $\Psi_\gamma(z, \eta, \delta) \in \mathcal{K}(\delta)^{p+1 \times 1}$, then from (28) we have $u \in \mathcal{K}^{m \times 1}(\delta)$, which implies the causal estimation of the unknown inputs.

\textbf{Theorem 2} indicates that the causality of the estimation for the unknown inputs depends on the matrix $\bar{\Gamma}_\gamma \in \mathcal{K}^m \times m(\delta)$.

If this matrix is not unimodular over $\mathcal{K}(\delta)$, then the estimation of $u$ might be non-causal. In order to take into account the non-causal estimation, let us introduce the forward shift operator $\nabla$ such that $\nabla x(t) = x(t + \tau)$.

Denote $\bar{\mathcal{K}}$ as the field of functions of finite number of variables from $\{x_i(t - j \tau), i \in [1, n], j \in [-s, s]\}$, then one can define the set $\bar{\mathcal{K}}^{n \times 1}(\delta, \nabla)$ whose entry is of the following form:

$$a(\delta, \nabla) = \bar{a}_{r_a}(t) + \ldots + \bar{a}_1(t) \nabla + a_0(t) + a_1(t) \delta + \ldots + a_{r_a}(t) \delta^+ \tag{25}$$

where all the coefficients belong to $\bar{\mathcal{K}}$. Then, for $\mathcal{K}(\delta, \nabla)$, the addition is as usual, but the multiplication is given by the following relation:

$$a(\delta, \nabla)[b(\delta, \nabla)] = \sum_{i=0}^{r_a} \sum_{j=0}^{r_b} a_i b_j \delta^{i+j} + \sum_{i=0}^{r_a} \sum_{j=1}^{r_b} a_i b_j \delta^{i+j} + \sum_{i=1}^{r_b} b_i \nabla a_i \delta^{i+j} \nabla \tag{26}$$

Finally, we have the following result.

\textbf{Lemma 1:} Under Assumption 1, if $\bar{\Gamma}_\gamma \in \mathcal{K}^m \times m(\delta)$ is unimodular over $\mathcal{K}(\delta, \nabla)$, then the unknown input $u$ of the system (6) can be at least non-causally estimated.

The proof of this lemma is quite similar to that of Theorem 2, thus the following will just explain the main difference. If the matrix $\bar{\Gamma}_\gamma$ is unimodular over $\mathcal{K}(\delta, \nabla)$, i.e., $\bar{\Gamma}_\gamma^{-1} \in \mathcal{K}^m \times m(\delta, \nabla)$, then the causality of the estimation for the unknown input $u$ depends on the power of the terms $\Gamma_\gamma^{-1}(z, \eta, \delta, \nabla)$, $Q_\gamma(z, \eta, \delta, \nabla)$, $\mathcal{H}_\gamma(z, \eta, \delta)$ in (20). However, the unknown input $u$ of the system (6) can be at least non-causally estimated.

\section{C. Observer design}

As we have explained, the estimation of the unknown input is based on the finite-time estimation of the state variable $z$ and their derivatives, which is also used to estimate the internal dynamics $\eta$. In what follows, we use the high order
sliding mode observer [12] to estimate $z$ and its derivatives:

$$
\begin{align*}
\dot{\hat{z}}_{i,j+1} &= \dot{\hat{z}}_{i,j} - \lambda_{i,j} |\dot{\hat{z}}_{i,j} - z_{i,j}|^{\varphi_i} \text{sign}(\dot{\hat{z}}_{i,j} - z_{i,j}) \\
\dot{\hat{z}}_{i,j+2} &= -\lambda_{i,j+1} |\dot{\hat{z}}_{i,j+1} - \dot{\hat{z}}_{i,j}|^{\varphi_j} \text{sign}(\dot{\hat{z}}_{i,j+1} - \dot{\hat{z}}_{i,j}) \\
\hat{z}_i &= \hat{z}_i, \theta_i \\
\hat{z}_{i,\theta_i-1} &= \hat{z}_i, \theta_i - \hat{z}_i, \theta_i - 2 \text{sign}(\hat{z}_i, \theta_i - 1 - \hat{z}_i, \theta_i - 2) \\
\hat{z}_i, \theta_i &= -\lambda_{i,\theta_i} \text{sign}(\hat{z}_i, \theta_i - \hat{z}_i, \theta_i - 1)
\end{align*}
$$

(27)

The convergence of the above observer has already been demonstrated in [12]. It has been shown that, if the $r$th ($r$ is the relative degree) derivative of the output has a Lipschitz constant $L > 0$ and by choosing $\lambda > L$, then we have $\hat{\hat{z}} = z$ after a finite time $T$. Then, the internal dynamics $\eta$ can be estimated by using the estimator (23). After $\eta$ was converged, the unknown input can be estimated from the following algebraic equation:

$$
\dot{\hat{u}} = (\hat{\Gamma}_\gamma)^{-1} \hat{Q}_\gamma (H_\gamma (\hat{\hat{z}}) - \Psi_\gamma (\hat{\hat{z}}, \hat{\eta}, \delta)).
$$

(28)

where $\hat{\Gamma}_\gamma = \Gamma_\gamma (\hat{\hat{z}}, \hat{\eta}, \delta)$ and $\hat{Q}_\gamma = Q_\gamma (\hat{\eta}, \hat{\hat{z}}, \delta)$ and the matrices $H_\gamma$, $\Psi_\gamma$ are given in the equation (20).

IV. NUMERICAL EXAMPLE

Let us consider the nonlinear time delay system:

$$
\begin{align*}
\dot{\xi}_1 &= \xi_2 \\
\dot{\xi}_2 &= -\xi_2 - \delta \xi_1 + u_1 \\
\dot{\xi}_3 &= -\xi_3 + 4 + cos(\frac{\xi_1}{100})u_1 \\
\dot{\xi}_4 &= -\xi_1,\xi_2 + (\frac{1}{1 - \delta \xi_5})u_2 \\
\dot{\xi}_5 &= -\xi_5 + sin(\frac{\xi_1}{50})u_2 \\
y_1 &= \xi_1, y_2 = \xi_3 , \quad \text{with} \quad \delta \xi_5 \neq 1.
\end{align*}
$$

(29)

Following the first step of the algorithm proposed in [22], we obtain

$$
\Gamma_\gamma = \begin{pmatrix}
1 & 0 \\
\cos(\frac{\delta \xi_5}{100}) & 0
\end{pmatrix}
$$

and this matrix is not full rank ($\text{rank}_{K(\delta)} \Gamma_\gamma = 1 < m$), then the unknown input $u_2$ cannot be estimated. To solve this problem, we have to used one more step the algorithm given in [22] to generate a new virtual output. Thus, we find

$$
\hat{y}_3 = \xi_4 = y_2 + \hat{y}_1 + \hat{y}_1 + \delta \hat{y}_1 \cos(\frac{\delta y_1}{100})
$$

which is a combination of the outputs and their derivatives and gives the reconstruction of $\xi_4$, only with known variables at the previous step. Thus, the internal dynamics is reduced to $\xi_5$. From section III, the system is rewritten as follows:

$$
\begin{align*}
\dot{\hat{z}}_1 &= z_2 \\
\dot{\hat{z}}_2 &= -z_2 - \delta z_1 + u_1 \\
\dot{\hat{z}}_3 &= z_3 + 4 + cos(\frac{\delta \xi_1}{100})u_1 \\
\dot{\hat{z}}_4 &= -z_1, z_2 + (\frac{1}{1 - \delta \eta})u_2 \\
\hat{\eta} &= -\eta + sin(\frac{\delta \xi_1}{50})u_2 \\
y_1 &= z_1, y_2 = z_3, \hat{y} = z_4, \delta \eta \neq 1.
\end{align*}
$$

(30)

The matrix $\Gamma_\gamma$ is directly obtained as:

$$
\Gamma_\gamma = \begin{pmatrix}
1 & 0 \\
\cos(\frac{\delta \xi_1}{100}) & 0
\end{pmatrix}
$$

(31)

It can be clearly seen that $\text{rank}_{K(\delta)} \Gamma_\gamma = m = 2$ then we can choose the matrix $Q_\gamma$ as

$$
Q_\gamma = \begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
$$

(32)

this gives the unimodular matrix:

$$
\Gamma_\gamma = \begin{pmatrix}
\cos(\frac{\delta \xi_1}{100}) & 0 \\
0 & 1 - \delta \eta
\end{pmatrix}
$$

(33)

Then, the unknown inputs are given by:

$$
\begin{align*}
u_1 &= \frac{1}{\cos(\frac{\delta \xi_1}{100})}[\hat{z}_3 + z_3 - z_4] \\
u_2 &= (1 - \delta \eta)[\hat{z}_4 + z_1 z_2]
\end{align*}
$$

(34)

where $u_2$ is, unfortunately, a function of the internal dynamics governed by:

$$
\eta = -\eta + sin(\frac{\delta \xi_1}{50})[(1 - \delta \eta)(\hat{z}_4 + z_1 z_2)]
$$

(35)

Consequently, $u$ can be estimated only after that $\hat{\eta}$ converges to $\eta$. As it can be seen the internal dynamic is a function of the states and their derivatives, which are estimated using the following High-Order Sliding Mode observer:

$$
\begin{align*}
\dot{\hat{z}}_{1,0} &= \hat{z}_{1,1} - \lambda_{1,0} |\hat{z}_{1,0} - y_1|^2 \text{sign}(\hat{z}_{1,0} - y_1) \\
\dot{\hat{z}}_{1,1} &= \hat{z}_{1,2} - \lambda_{1,1} |\hat{z}_{1,1} - \hat{z}_{1,0}|^2 \text{sign}(\hat{z}_{1,1} - \hat{z}_{1,0}) \\
\dot{\hat{z}}_{1,2} &= -\lambda_{1,2} \text{sign}(\hat{z}_{1,2} - \hat{z}_{1,1}) \\
\dot{\hat{z}}_{2,0} &= \hat{z}_{2,1} - \lambda_{2,0} |\hat{z}_{2,0} - y_2|^2 \text{sign}(\hat{z}_{2,0} - y_2) \\
\dot{\hat{z}}_{2,1} &= -\lambda_{2,1} \text{sign}(\hat{z}_{2,1} - \hat{z}_{2,0}) \\
\dot{\hat{z}}_{3,0} &= \hat{z}_{3,1} - \lambda_{3,0} |\hat{z}_{3,0} - y_3|^2 \text{sign}(\hat{z}_{3,0} - y_3) \\
\dot{\hat{z}}_{3,1} &= -\lambda_{3,1} \text{sign}(\hat{z}_{3,1} - \hat{z}_{3,0})
\end{align*}
$$

(36)

As (36) is a finite time observer, there exists a finite time $T$ such that for all $t > T$ we have $\hat{z}_{1,0} = y_1$, $\hat{z}_{1,1} = \hat{y}_1$, $\hat{z}_{1,2} = \hat{y}_1$, $\hat{z}_{2,0} = y_2$, and $\hat{z}_{2,1} = \hat{y}_2$ thus $\hat{y}_3 = z_4 = \hat{z}_{3,0}$.  

473
Now, we design the following estimator for the internal dynamic:
\[
\dot{\hat{\eta}} = -\hat{\eta} + \sin(\delta \hat{z}_1(\frac{50}{23}))[1 - \delta \hat{\eta}](\dot{\hat{z}}_4 + \hat{z}_1 \hat{z}_2)]
\]  

(37)

Choosing a Lyapunov function
\[
V(e_\eta) = \frac{1}{2} e_\eta^T e_\eta
\]

(38)

with \(e_\eta = \eta - \hat{\eta}\), we obtain:
\[
\dot{V} = -(\eta - \hat{\eta})^2 - (\eta - \hat{\eta})[\delta(\eta - \hat{\eta})f(e) + f(e)]
\]

(39)

with \(f(e) = \sin(\frac{2 \pi}{50}(\hat{z}_4 + \hat{z}_1 \hat{z}_2)) - \sin(\frac{2 \pi}{50}(\hat{z}_1 \hat{z}_2))\).

When \(t > T\) we have \(e = 0\), i.e. \(f(e) = 0\), so \(\dot{V} = -(\eta - \hat{\eta})^2 < 0\), thus Assumption 1 is verified. Then the unknown inputs can be causally estimated from the following algebraic relation:
\[
\begin{align*}
\dot{\hat{u}}_1 &= \frac{1}{\cos(\frac{2 \pi}{50})}[\hat{z}_3 + \hat{z}_3 - \hat{z}_4] \\
\dot{\hat{u}}_2 &= (1 - \delta \hat{\eta})(\hat{z}_4 + \hat{z}_1 \hat{z}_2)
\end{align*}
\]

(40)

For the simulation, the initial conditions for the studied example are chosen as \(z_0 = (1, 1, 3, 5, 1)^T\), and the initial conditions for the observer are fixed to zero. The gains for the high-order sliding mode observer are \(\lambda_{1,0} = 5, \lambda_{1,1} = 2, \lambda_{1,2} = 1.9, \lambda_{2,0} = 2.5, \lambda_{2,1} = 3, \lambda_{3,0} = 6, \lambda_{3,1} = 5\). The simulation results are depicted in Fig. 1, where it clearly shows the convergence of the estimation of the unknown input.

![Fig. 1. Estimation of unknown inputs \(u_1\) and \(u_2\).](image)

V. CONCLUSION

This paper proposed a solution based on high-order sliding mode observer to solve the left inversion problem of nonlinear time delay system with unknown inputs. The delay is considered as constant, and under the assumption on the internal dynamics, the left invertibility problem can be solved. The problem of causality and non-causality for the estimation of the unknown inputs are discussed.