Homogenization in reaction-diffusion PDEs under space and time-dependent heterogeneities

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Abstract—We study spatial homogenization of nonlinear reaction-diffusion PDEs subject to a class of spatially and temporally-varying heterogeneities. We show that an incremental passivity property in the nominal reaction dynamics is fundamental in guaranteeing homogenization of trajectories. Our distributed control achieves homogenization by defining an internal model subsystem for each output corresponding to the respective heterogeneity. We illustrate our algorithm on a system with bistable dynamics. The main contribution of the paper is a constructive approach to guarantee spatial homogenization under heterogeneities.

I. INTRODUCTION

Spatially distributed models with diffusive coupling are crucial to understanding the dynamical behavior of a range of engineering and biological systems. This form of coupling encompasses, among others, feedback laws for coordination of multi-agent systems, electromechanical coupling of synchronous machines in power systems, and local update laws in distributed agreement algorithms. Synchronization of diffusively-coupled nonlinear systems is an active and rich research area [1]; conversely, developing conditions that rule out synchrony is also important, as they facilitate study of spatial pattern formation. One of the major ideas behind pattern formation in cells and organisms is based on diffusion-driven instability [2], [3], which occurs when higher-order spatial modes in a reaction-diffusion PDE are destabilized by diffusion [4]–[8].

Several works in the literature study the case of static interconnections between nodes in full state models [9]–[15] or phase variables in phase coupled oscillator models [16]–[19]. Common to much of the literature is the assumption that the agents to be synchronized are homogeneous with identical dynamics, and are furthermore not subject to disturbances.

However, recent work has considered synchronization and consensus in the presence of exogenous inputs. In [20], the authors addressed robust dynamic average consensus (DAC), in which partial model information about a broad class of time-varying inputs enabled exact tracking of the average of the inputs through the use of the internal model principle [21] and the structure of the proportional-integral average consensus estimator formulated in [22]. The internal model principle has been useful in establishing necessary and sufficient conditions for output regulation [23] and synchronization [24]–[26]. Reference [27] proposed internal model control strategies in which controllers were placed on the edges of the interconnection graph to achieve output synchronization under time-varying disturbances. Recent work has also addressed robust synchronization in cyclic feedback systems [28] and in the presence of structured uncertainties [29].

Spatial homogenization of reaction-diffusion PDEs has also been studied in the literature. In [9], the author gave a Lyapunov inequality for the Jacobian of the reaction term, significantly reducing conservatism of dominant approaches making use of global Lipschitz bounds [30]. However, the problem of spatial homogenization of reaction-diffusion PDEs subject to heterogeneities has not yet been addressed.

In [31], we considered synchronization of nonlinear systems satisfying an incremental passivity property and subject to a class of disturbance inputs including constants and sinusoids. Building on the robust DAC estimator in [20], we proposed a distributed control law that achieves output synchronization under disturbances by defining an internal model subsystem at each node corresponding to the disturbance inputs.

In this paper, we extend the results in [31] to the PDE case and present a distributed control law that achieves spatial homogenization in reaction-diffusion PDEs in the presence of spatially and temporally-varying heterogeneities. Using the results in [31] as a starting point, we develop a similar controller for each output of the reaction-diffusion system corresponding to each heterogeneity. Our controller applies to systems with multiple input and output channels and allows non-identical heterogeneities to enter each channel.

The rest of the paper is organized as follows. Section II reviews output synchronization of diffusively-coupled networks with incrementally passive systems under input disturbances. Our main result on spatial homogenization under heterogeneities is presented Section III. In Section IV, we illustrate the effectiveness of our control law on a model with bistable dynamics. Conclusions and future work are discussed in Section V.

Notation: Let $1_N$ be the $N \times 1$ vector with all entries 1. Let $0_N$ be the $N \times 1$ vector with all entries 0. Let the transpose of a real matrix $A$ be denoted by $A^T$. Let the notation $M = \text{blkdiag}(M_1, \ldots, M_p)$ denote the block diagonal matrix $M$ with matrices $M_i$, $i = 1, \ldots, p$, along the diagonal.

II. SYNCHRONIZATION OF COMPARTMENTAL SYSTEMS OF ODEs

Consider a collection of $N$ dynamical systems $H_i$, $i = 1, \ldots, N$, defined by:

$$H_i: \begin{align*}
\dot{x}_i &= f(x_i) + g(x_i)u_i \quad i = 1, \ldots, N \quad (1) \\
y_i &= h(x_i). \quad (2)
\end{align*}$$

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in which \( x_i \in \mathbb{R}^n, \ u_i \in \mathbb{R}^p, \ y_i \in \mathbb{R}^p, \) and \( f(\cdot) : \mathbb{R}^n \to \mathbb{R}^n, \ g(\cdot) : \mathbb{R}^n \to \mathbb{R}^{n \times p}, \) and \( h(\cdot) : \mathbb{R}^n \to \mathbb{R}^p \) are continuously differentiable maps. \( H_i \) is said to satisfy an incremental output-feedback passivity property \([12], [14]\) if there exists a positive definite storage function \( S : \mathbb{R}^n \to \mathbb{R} \) such that for any two solution trajectories \( x_{i}(t) \) and \( x_{i}(t) \) of \( H_i \) with input-output pairs \( u_{i0}(t), y_{i0}(t) \) and \( u_{i0}(t), y_{i0}(t) \):

\[
\frac{d}{dt}S(\delta x) = S(\delta x) \leq \theta \delta y^T \delta y + \delta u^T \delta y, \tag{3}
\]

with \( \delta x = x_{i}(t) - x_{i}(t), \ \delta y = y_{i}(t) - y_{i}(t), \) and \( \delta y = y_{i}(t) - y_{i}(t). \) When \( \theta \leq 0, \ H_i \) is called incrementally passive, and when \( \theta < 0, \ H_i \) is called output-striktly incrementally passive.

In \([31]\), we considered the scenario with \( p = 1 \) where the inputs \( u_i \) were subjected to a class of unknown \( \phi_i(t) \in \mathbb{R}, \) i.e.,

\[
u_i = \ddot{u}_i + \phi_i, \tag{4}\]

in which each disturbance \( \phi_i(t) \) could be characterized by

\[
\dot{\xi}_i = A\xi_i, \quad \xi_i(0) \in \mathbb{R}^n \tag{5}
\]

\[
\phi_i = C\xi_i, \tag{6}
\]

with \( A = -A^T \in \mathbb{R}^{n \times n} \) and the pair \((A, C)\) observable. This class includes constant as well as sinusoidal disturbances. We then proposed the controller

\[
u_i = -\sum_{(i,j) \in E_i} p_{ij}(y_j - y_i) - \sum_{(i,j) \in E_i} n_{ij}(\eta_i - \eta_j), \tag{7}\]

with \( E, E_i \) defining the edge sets of undirected graphs \( G, \ G_i \) representing coupling between terms \( y_i \) and \( \eta_i \) specified below in (8), respectively, where \( p_{ij} = p_{ji} > 0 \) if \((i, j) \in E_j\), \( p_{ij} = 0, (i, j) \notin E_j, \) \( n_{ij} = n_{ji} > 0 \) if \((i, j) \in E_i, \) and \( n_{ij} = 0, (i, j) \notin E_i. \)

Corresponding to each subsystem \( H_i, \) we defined the internal model system \( G_i \) given by

\[
G_i : \begin{align*}
\dot{\xi}_i &= A\xi_i + B_i \sum_{(i,j) \in E_i} n_{ij}(y_j - y_i) \tag{8} \\
\eta_i &= B_i^T \xi_i, \tag{9}
\end{align*}
\]

where \((A, B_i^T)\) is designed to be observable and \( \xi_i(0) \) the initial condition of \( \xi_i \) may be arbitrarily chosen. Note that since \( A = -A^T \) and \( G_i \) is a linear system, it is straightforward to show that \( G_i \) is passive and thus incrementally passive from \( \sum_{(i,j) \in E_i} n_{ij}(y_j - y_i), \) to \( \eta_i. \)

Using the control in (7), we proved the following theorem to guarantee output synchronization of the subsystems \( H_i: \)

**Theorem 1:** \([31]\) Consider the nonlinear systems \( H_i \) in (1) and (2) satisfying (3) with the input given in (4), (7), (8) and (9). Suppose that \( \lambda_2 - \theta > 0 \) and that \( G_i \) is connected. If the solutions are bounded, then the outputs \( y_i \) synchronize asymptomatically:

\[
\lim_{t \to \infty} \left| y_i(t) - \frac{1}{N} \mathbf{1}_N y(t) \right| = 0, \quad \forall i \in \{1, \ldots, N\}. \tag{10}
\]

**III. SPATIAL HOMOGENIZATION OF REACTION-DIFFUSION PDEs**

In this section, we formulate the problem of spatial homogenization for systems of reaction-diffusion PDEs, which is analogous to the problem of synchronization of like components in compartmental systems of ODEs.

Let \( \Omega \) be a bounded and connected domain in \( \mathbb{R}^r \) with smooth boundary \( \partial \Omega, \) and consider the PDE:

\[
\frac{\partial x(t, \chi)}{\partial t} = f(x(t, \chi)) + \sum_{\ell=1}^p g_{\ell}(x(t, \chi)) u_{\ell}(t, \chi), \tag{11}
\]

\[
y(t, \chi) = h(x(t, \chi)), \quad \ell = 1, \ldots, p, \tag{12}
\]

where \( \chi \in \Omega \) is the spatial variable, \( x(t, \chi) \in \mathbb{R}^n \) is the state variable with initial condition \( x(0, \chi) = x_0(\chi), \) \( f(\cdot) : \mathbb{R}^n \to \mathbb{R}^n, \) \( g(\cdot) = [g_1 \cdots g_p] : \mathbb{R}^n \to \mathbb{R}^{n \times p}, \) and \( h(\cdot) = [h_1^T \cdots h_p^T] : \mathbb{R}^n \to \mathbb{R}^p \) are continuously differentiable, and the input \( u_\ell \) consists of differential operators to be specified. We assume Neumann boundary conditions:

\[
\nabla x(t, \chi) \cdot \hat{n}(\chi) = 0 \quad \forall \chi \in \partial \Omega, \ \forall t \geq 0, \quad i = 1, \ldots, n \tag{13}
\]

where \( \nabla \) represents the gradient with respect to the spatial variable \( \chi. \) ”\( \cdot \) “ is the inner product in \( \mathbb{R}^r, \) \( x(\chi) \) denotes the \( \ell \)th entry of the vector \( x(t, \chi) \) and \( \hat{n}(\chi) \) is a vector normal to the boundary \( \partial \Omega. \)

We consider the scenario where each input \( u_\ell(t, \chi) \) is subject to a class of unknown heterogeneities \( \phi_\ell(t, \chi), \) i.e.,

\[
u_\ell(t, \chi) = \ddot{u}_\ell(t, \chi) + \phi_\ell(t, \chi). \tag{14}\]

We assume that the heterogeneity \( \phi_\ell(t, \chi) = [\phi_1(t, \chi) \ldots \phi_p(t, \chi)]^T \) can be characterized by

\[
\dot{\xi}_\ell(t, \chi) = A_\ell \xi_\ell(t, \chi) \tag{15}
\]

\[
\phi_\ell(t, \chi) = C_\ell \xi_\ell(t, \chi), \quad \ell = 1, \ldots, p, \tag{16}
\]

in which \( A_\ell \in \mathbb{R}^{n \times n} \) satisfies \( A_\ell = -A_\ell^T, \ C_\ell \in \mathbb{R}^n, \) the pair \((A_\ell, C_\ell)\) is observable, and the initial condition \( \xi_\ell(0, \chi) \) may be arbitrarily chosen in \( L_2(\Omega). \) We assume that the matrix \( A_\ell \) is available.

In relation to synchronization of each like output \( y_{ik} \) across compartments, we derive a control law that guarantees homogenization of each output \( y_{i}(t, \chi) \) across the spatial domain \( \Omega, \) whose topology is the spatial continuum analogue to the topology of the interconnection graph between compartments.

**Main Result**

We seek to design the control \( u^*(t, \chi) \) such that the outputs \( y(t, \chi) = [y_1(t, \chi) \ldots y_p(t, \chi)]^T \) are homogenized spatially. We consider the following design of \( u^*(t, \chi): \)

\[
u^*(t, \chi) = \nabla \cdot (p(\chi)\nabla y(t, \chi)) - \nabla \cdot (n(\chi)\nabla v(t, \chi)), \tag{17}\]

in which \( \nabla \cdot \) is the divergence operator, \( \nabla \cdot (p(\chi)\nabla y(t, \chi)) \) is the vector representation of the diffusion operator \( \nabla \cdot (n(\chi)\nabla \cdot) \) applied to each component \( v(t, \chi) \) of \( v(t, \chi): \)

\[
\nabla \cdot (n(\chi)\nabla v(t, \chi)) = ([\nabla \cdot (n(\chi)\nabla v_1(t, \chi)) \ldots \nabla \cdot (n(\chi)\nabla v_p(t, \chi))])^T, \tag{18}\]
\( p(\chi), n(\chi) \in \mathbb{R} \) are diffusion coefficients with \( p(\chi) > 0, n(\chi) > 0 \), and \( \eta \) is the output of the internal model system \( G \):

\[
G : \begin{align*}
\frac{\partial \zeta(t, \chi)}{\partial t} &= A \zeta(t, \chi) + B \nabla \cdot (n(\chi) \nabla y(t, \chi)) \\
\eta(t, \chi) &= B^T \zeta(t, \chi),
\end{align*}
\]

where \((A, B^T)\) is designed to be observable with \( B \in \mathbb{R}^{dx \times p} \), and the initial condition \( \zeta(0, \chi) \) may be arbitrarily chosen in \( L^2(\Omega) \).

Define:

\[
\bar{x}(t) := \frac{1}{|\Omega|} \int_{\Omega} x(\chi) d\chi, \quad \bar{x}(t) := x(\chi) - \bar{x}(t)
\]

\[
\tilde{y}(t, \chi) := y(t, \chi) - \bar{y}(t).
\]

In proving our main result, we will make use of a Lemma following from the Poincaré principle [32, Equation (1.37)]:

**Lemma 1:** Let \( \lambda_2 \) denote the second smallest of the Neumann eigenvalues 0 \( = \lambda_1 \leq \lambda_2 \leq \cdots \) of the operator \( L = -\nabla \cdot (d(\chi) \nabla \cdot (\cdot)) \) with \( d(\chi) > 0 \) on the connected, bounded domain \( \Omega \) with smooth boundary \( \partial \Omega \) and spatial variable \( \chi \in \Omega \). Let \( u(\chi, t) \) be a function not identically zero in \( L^2(\Omega) \) with derivatives \( \frac{\partial u}{\partial \chi} \in L^2(\Omega) \) that satisfies the Neumann boundary condition \( \nabla u(\chi, t) \cdot n(\chi) = 0 \) and \( \int_{\Omega} u(\chi, t) d\chi \neq 0 \).

Then the following inequality holds:

\[
\int_{\Omega} \nabla u(\chi, t) \cdot (d(\chi) \nabla u(\chi, t)) d\chi \geq \lambda_2 \int_{\Omega} u(\chi, t)^2 d\chi.
\]

\[
\Box
\]

In Theorem 2 below, we give conditions that guarantee the following output synchronization property:

\[
\lim_{t \to \infty} \int_{\Omega} |\tilde{y}(t, \chi)|^2 d\chi = 0,
\]

where \(| \cdot |\) denotes the Euclidean norm.

**Theorem 2:** Consider the system (11)-(12) with the boundary condition in (13) on a connected, bounded domain \( \Omega \), with the input given in (14), (17), (19)-(20) and \( B^T \) \( = \) \( B \in \mathbb{R}^{dx \times p} \). Suppose there exists a storage function for (11)-(12) satisfying (3) with \( \lambda_2 - \theta > 0 \). Then for every bounded classical solution\(^1\), the outputs synchronize in the sense of (23).\( \Box \)

Theorem 2 applies to classical solutions that exist for all \( t \geq 0 \). Results on the existence of classical solutions to reaction-diffusion PDEs can be found in [34]-[36].

**Proof of Theorem 2:** We first consider the function:

\[
V = \frac{1}{2|\Omega|} \int_{\Omega} \int_{\Omega} S(x(t, \chi_a) - x(t, \chi_b)) d\chi_a d\chi_b.
\]

Note that (11)-(12), without the coupling term (14), is identical to (1)-(2). Therefore, using (3), we have:

\[
\dot{V} < \frac{1}{2|\Omega|} \int_{\Omega} \int_{\Omega} \theta(y(t, \chi_a) - y(t, \chi_b)) \cdot (y(t, \chi_a) - y(t, \chi_b))
\]

\[
+ (u(t, \chi_a) - u(t, \chi_b)) \cdot (y(t, \chi_a) - y(t, \chi_b)) d\chi_a d\chi_b.
\]

Defining \( \bar{u}(t) := \frac{1}{|\Omega|} \int_{\Omega} u(t, \chi) d\chi \) and \( \bar{y}(t) := \frac{1}{|\Omega|} \int_{\Omega} y(t, \chi) d\chi \), we have:

\[
\int_{\Omega} \int_{\Omega} \left((u(t, \chi_a) - u(t, \chi_b)) \cdot (y(t, \chi_a) - y(t, \chi_b))
\]

\[
+ (u(t, \chi_a) - u(t, \chi_b)) \cdot (y(t, \chi_a) - y(t, \chi_b)) d\chi_a d\chi_b.
\]

\[
\Box
\]

\(^1\)A solution of a PDE of order \( k \) is said to be classical if it is at least \( k \) times continuously differentiable so that all derivatives that appear in the PDE exist and are continuous [33].
Applying the Divergence Theorem and noting that \( \nabla \tilde{y}_\ell(t, \chi) \cdot \hat{n}(\chi) = 0 \) for \( \chi \in \partial \Omega \) from the boundary condition (13), we have:
\[
\int_\Omega \nabla \cdot (\tilde{y}_\ell(t, \chi)p(\chi)\nabla \tilde{y}_\ell(t, \chi))d\chi \\
= \int_\Omega \tilde{y}_\ell(t, \chi)p(\chi)\nabla \tilde{y}_\ell(t, \chi) \cdot \hat{n}(\chi)dS = 0,
\]
verifying (35).

Moreover, because \( \int_\Omega \tilde{y}_\ell(t, \chi)d\chi = 0 \), it follows from Lemma 1 that:
\[
\int_\Omega \tilde{y}_\ell(t, \chi)d\chi \geq \lambda_2 \int_\Omega \tilde{y}_\ell(t, \chi)^2d\chi.
\]
Substituting in (34) and using the fact that \( \int_\Omega \tilde{y}_\ell(t, \chi)d\chi = 0 \), we have:
\[
V \leq \int_\Omega (\lambda_2 - \theta)\tilde{y}(t, \chi) \cdot \tilde{y}(t, \chi) \\
+ \sum_{i=1}^p \tilde{y}_i(t, \chi)\left( \frac{d}{dt} - \nabla \cdot (n(\chi)\nabla \eta(\chi)) \right)d\chi,
\]
where \( \tilde{y}_i(t, \chi) = \phi_i(t, \chi) - \tilde{y}_i(t, \chi) \).

We next consider the auxiliary system:
\[
\frac{d}{dt}\xi(t, \chi) = A\xi(t, \chi) \tag{41}
\]
\[
\tilde{y}(t, \chi) = C\xi(t, \chi). \tag{42}
\]
Noting that the eigenfunctions of the elliptic operator \( \nabla \cdot (n(\chi)\nabla \cdot) \) form a complete orthogonal basis for \( L_2(\Omega) \), and that \( \int_\Omega \phi_i(t, \chi)d\chi = 0, \ell = 1, \ldots, p \), the initial conditions of \( \xi(t, \chi) \) may be chosen such that
\[
\nabla \cdot (n(\chi)\nabla \phi_i(0, \chi)) = \phi_i(0, \chi), \quad \ell = 1, \ldots, p \tag{43}
\]
and thus \( \nabla \cdot (n(\chi)\nabla \tilde{y}_i(t, \chi)) = \phi_i(t, \chi) \) for all \( \ell = 1, \ldots, p \).

We define
\[
\delta(t, \chi) := \xi(t, \chi) - \tilde{y}(t, \chi),
\]
and consider the following storage function:
\[
W = \frac{1}{2} \int_\Omega \delta(t, \chi) \cdot \dot{\delta}(t, \chi)d\chi,
\]
Differentiating \( W \) with respect to time, we obtain:
\[
W = \int_\Omega \delta(t, \chi) \cdot C(\nabla \cdot (n(\chi)\nabla \tilde{y}(t, \chi)))d\chi \\
= \int_\Omega (\eta(t, \chi) - \dot{\delta}(t, \chi)) \cdot (\nabla \cdot (n(\chi)\nabla \tilde{y}(t, \chi)))d\chi.
\]
The sum \( Z = V + W \) yields
\[
\dot{Z} = \dot{V} + \dot{W} \\
\leq \int_\Omega -(\lambda_2 - \theta)\tilde{y}(t, \chi) \cdot \tilde{y}(t, \chi) \\
+ \sum_{i=1}^p \left[ \tilde{y}_i(t, \chi) \left( \frac{d}{dt} - \nabla \cdot (n(\chi)\nabla \eta(\chi)) \right) \right]d\chi,
\]
where we now make use of the self-adjointness property of the diffusion operator, apply the identity in (36) with \( f = \tilde{y}(t, \chi) \) and \( F = n(\chi)\nabla \eta(t, \chi) \), and get
\[
\int_\Omega \tilde{y}(t, \chi) \nabla \cdot (n(\chi)\nabla \eta(t, \chi))d\chi \\
= -\int_\Omega n(\chi)\nabla \tilde{y}(t, \chi) \cdot \nabla \tilde{y}(t, \chi)d\chi \\
= \int_\Omega \eta(t, \chi) \nabla \cdot (n(\chi)\nabla \tilde{y}(t, \chi))d\chi.
\]
where the second equality follows by applying the identity with \( f = \eta(t, \chi) \) and \( F = n(\chi)\nabla \tilde{y}(t, \chi) \). Similarly, we have:
\[
\int_\Omega \frac{d}{dt}(t, \chi) \cdot (n(\chi)\nabla \tilde{y}(t, \chi))d\chi \\
= \int_\Omega \tilde{y}(t, \chi) \nabla \cdot (n(\chi)\nabla \tilde{y}(t, \chi))d\chi.
\]
Substituting (53) and (54) into (50), we then have:
\[
Z \leq \int_\Omega -(\lambda_2 - \theta)\tilde{y}(t, \chi) \cdot \tilde{y}(t, \chi) \\
+ \sum_{i=1}^p \tilde{y}_i(t, \chi) \left( \frac{d}{dt} - \nabla \cdot (n(\chi)\nabla \tilde{y}(t, \chi)) \right)\left( \frac{d}{dt} - \nabla \cdot (n(\chi)\nabla \tilde{y}(t, \chi)) \right)d\chi.
\]
Choosing \( \xi(0, \chi) \) such that (43) is satisfied with \( \epsilon := \lambda_2 - \theta \), we have:
\[
\dot{Z} \leq \int_\Omega -(\lambda_2 - \theta)\tilde{y}(t, \chi) \cdot \tilde{y}(t, \chi) \\
- \epsilon \int_\Omega |\tilde{y}(t, \chi)|^2d\chi = -\epsilon \Omega(t).
\]
This implies that \( \lim_{t \to -\infty} \int_0^t \Omega(t)dt \) exists and is bounded. Since \( \Omega(t) \) is also bounded, it follows from Barbalat’s Lemma [37] that \( \Omega(t) \to 0 \) as \( t \to \infty \), which proves (23). \( \square \)

**Remark 1:** When \( A \) and \( C \) have block diagonal structure, we may choose \( B \neq C \). Suppose \( A = \text{blkdiag}(A_1, \ldots, A_p) \) with \( A_k \in \mathbb{R}^{d_k \times d_k} \) and \( \sum_{k=1}^p d_k = d \), \( C \in \mathbb{R}^{p \times d} \) with \( C = \text{blkdiag}(C_1, \ldots, C_p) \) with \( C_k \in \mathbb{R}^{d \times d_k} \), and the pairs \( (A_k, C_k) \) observable. Then \( B \) in (19)-(20) may be chosen such that \( B = \text{blockdiag}(B_1, \ldots, B_p) \) and \( B_2 \) is \( \mathbb{R}^{d \times d_k} \) with the pairs \( (A_k, B_k) \) observable. To address the fact that each \( B_k \) differs from \( C \), we introduce the auxiliary systems
\[
\frac{d}{dt}z(t, \chi) = Az(t, \chi), \tag{58}
\]
\[
\psi_k(t, \chi) = B_k^* z(t, \chi), \tag{59}
\]
with \( z(0, \chi) \) in \( L_2(\Omega) \), and define \( \psi = [\psi_1^T, \ldots, \psi_p^T]^T \).

We show that an appropriate choice of \( z(0, \chi) \) implies \( \psi = \phi \), and thus guarantees
\[
\nabla \cdot (n(\chi)\nabla \psi_k(t, \chi)) = \phi_k(t, \chi) \quad \text{for all } k = 1, \ldots, p. \tag{60}
\]
We choose \( z(0) = O_B^* O_C \dot{\xi}(0) \), where \( O_B \) is the block diagonal matrix of observability matrices corresponding to the pairs \( (A_k, B_k^*) \) and \( O_C \) is the block diagonal matrix of observability matrices corresponding to the pairs \( (A_k, C_k) \). Since \( z(0) = O_B^{-1} O_C \dot{\xi}(0) \), \( z(t) = O_B^{-1} O_C \dot{\xi}(t) \), which means \( O_B z(t) = O_C \dot{\xi}(t) \). Noting that the first row of each block \( k \) of \( O_B \) and \( O_C \) is \( B_k^* \) and \( C_k \), respectively, we have \( \psi = \text{blkdiag}[B_1^*, \ldots, B_p^*, C_1^*, \ldots, C_p^*] = \text{blkdiag}[C_1, \ldots, C_p] \xi$, $i = 1, \ldots, N$. We modify the differ-
ence \( \delta_i \) in (44) and (45) as \( \delta := \zeta - z \). The rest of the proof is identical and is omitted. □

IV. Numerical Example

In this section, we study a system with nominally bistable dynamics subject to heterogeneities, and demonstrate the effectiveness of the internal model approach in guaranteeing spatial homogenization.

Consider a system defined with the dynamics

\[
f(x) = x - x^3
\]

with \( y = x \) and \( g(x(t, \chi)) = 1 \). Note from (61) that each point in the space is a bistable system with stable equilibria at \( \pm 1 \) and a saddle point at 0. It can be shown that there exists a quadratic storage function such that the system satisfies the inequality (3) with \( \theta = -2 \) [38], and so the assumption of Theorem 2 is satisfied whenever \( \lambda_2 > 2 \).

We subject the system to a spatially and temporally-varying heterogeneity:

\[
\phi(t, \chi) = 10\sin(\pi t + 2\pi \chi), \quad \ell = 1, \ldots, 3.
\]

We consider the initial condition

\[
x(0, \chi) = 3.4\cos(1.3\pi \chi)\exp(-2\chi^2).
\]

We simulate the system on the domain \( \Omega = [0, 1] \). Figure 1 shows the evolution of \( x(t, \chi) \) in the absence of heterogeneities. In Figure 2, we show the trajectory of \( x(t, \chi) \) in the presence of heterogeneities without the internal model control. Under the internal model control with \( n(\chi) = 1 \) and \( \zeta(0, \chi) = 1 \), the trajectory of \( x(t, \chi) \) becomes spatially homogeneous (Figure 3), and effects due to the differences \( \phi(t, \chi) - \phi(t) \) are compensated. The remaining (spatially homogeneous) effect is due to \( \bar{\phi}(t) \), which is zero for (62); thus, the trajectory tends to an equilibrium value across the spatial and temporal dimensions. Figure 4 shows the output \( \eta(t, \chi) \) of the controller under the heterogeneity in (62).

We next modify \( \phi(t, \chi) \):

\[
\phi(t, \chi) = 10\sin(\pi t + 1.7\pi \chi), \quad \ell = 1, \ldots, 3,
\]

and simulate the system again with internal model control. Note that since \( \bar{\phi}(t) \) is no longer zero, the trajectory in Figure 5 exhibits spatially uniform oscillations.

V. Conclusion

We have designed a distributed control law that guarantees spatial homogeneity in incrementally-passive systems under a class of spatially and temporally-varying heterogeneities. Our controller handles heterogeneities that vary over input and output channels and does not require their initial conditions. In forthcoming work, we will demonstrate the use of adaptation of the diffusion coefficients as in [38] to reduce time to spatial homogeneity, and will address additional classes of heterogeneities.

REFERENCES

Fig. 4. Evolution of internal model controller $\eta(t,\chi)$ with initial condition $\eta(0,\chi) = 1$ and heterogeneity as in (62).

Fig. 5. Evolution of $x(t,\chi)$ as in (61) with initial condition $x(0,\chi) = 3.4 \cos(1.3\pi t) \exp(-2\chi^2)$ with heterogeneity as in (64) with internal model controller. Spatial homogenization occurs with spatially uniform oscillations.


