Unfalsified Adaptive Control with Reset and Bumpless Transfer

Sagar V. Patil, Yu-Chen Sung, Michael G. Safronov

Abstract—Adaptive switching control for time-varying plants based on the Battistelli-Hespanha-Mosca-Tesi reset mechanism is investigated. A new reset mechanism is designed and the switching algorithm is improved to admit bumpless transfer at switching times, and to relax several restrictive assumptions. Stability is proved given only that the adaptive stabilization problem is feasible, removing the assumption that the plant is linear and relaxing restrictions on controller state-initialization at switching times so as to allow bumpless transfer. An example is provided to support improved results.

I. INTRODUCTION

The hysteresis algorithm [1], the stability overlay algorithm [2] and the increasing cost level algorithm [3] are all examples of adaptive controller switching algorithms that can be interpreted within the theoretical framework of unfalsified adaptive control. But without modifications these algorithms are not suitable for controlling a time-varying plant. The problem is that they use cost functions with infinite memory which give equal importance to every element of the measured data no matter how old the data. In order to extend the results of unfalsified adaptive control to time-varying systems, one solution introduced in [5] is to use a non-monotone cost function with the fading memory data. In the same context, another novel scheme proposed in [7], [8] is to use the fading memory data to evaluate the monotone cost function over the most recent memory time-window. The length of memory time-window is decided adaptively by a so called reset condition. Both these approaches allow the supervisor to understand the most recent behavior of the time-varying system and, without waiting for too long, select an appropriate cost minimizing controller.

The contribution of the present paper is the relaxation of several restrictive assumptions in the recent work of Battistelli, Hespanha, Mosca and Tesi (BHMT) [7]. Among the assumptions that we relax are (i) the plant to be controlled is a strictly causal linear system of bounded order, (ii) each candidate controller is linear and (iii) the initial conditions of the candidate controllers are not re-initialized or adjusted at switching times. By discarding these assumptions this paper aims to improve the theoretical results in [7], so that the only assumption required to prove stability of the adaptive system is that the adaptive stabilization problem is feasible. A new and improved reset condition is proposed with the flexibility to handle undesired bumpy transients in the control signal. Hence the bumpless transfer technique of [6] based on slow-fast controller decomposition is implemented.

Sagar V. Patil, Yu-Chen Sung and Michael G. Safronov are with the Department of Electrical Engineering - Systems, University of Southern California, Los Angeles, CA 90089-2563, USA. E-mail: sagarvpa@usc.edu, yuchens@usc.edu and msafronov@usc.edu

II. PROBLEM FORMULATION

In this paper, we consider discrete time signals and systems. \( Z, Z^+ \) and \( \mathbb{R}_+ \) represent the sets of integers, non-negative integers and non-negative real numbers respectively. The absolute value, transpose, pseudo inverse and ceiling function are denoted by \(|·|\), \([·]'\), \([·]'^{-1}\) and \([·]\) respectively. For a real valued signal \( x \) defined on \( Z_+ \), the \( \lambda \) exponentially weighted infinity norm \( \|x\|_{\lambda,\infty} \) on the given interval \( [t_0,t] \) is defined as \( \|x\|_{\lambda,\infty} = \max_{\tau \in [t_0,t]} \lambda^{t-\tau} |x(\tau)| \) where \( \lambda \in (0,1) \), and the \( \ell_\infty \) norm is defined as \( \|x\|_{\ell,\infty} = \max_{\tau \in [t_0,t]} |x(\tau)| \). If \( \|x\|_{\lambda,\infty} < \infty \) for all \( t \in Z^+ \) then \( x \in \ell_\lambda\infty \). If \( \|x\|_{\ell,\infty} < \infty \) for all \( t \in Z^+ \) then \( x \in \ell_\infty \). If \( \exists \theta < \infty \) such that \( \|x\|_{\ell,\infty} < \theta \), \( \forall t \geq 0 \) then \( x \in \ell_{\infty} \) and \( x \) is called a stable signal. \( \ell_{\lambda,\infty} \) has the following properties.

\( a: \|x\|_{\lambda,\infty}(t) \leq \|x\|_{\ell,\infty} + \|x\|_{\lambda,\infty}(t-\tau) \lambda^{-t} \), \( \forall \tau \leq t \)

\( b: \|x\|_{\lambda,\infty}(t) \lambda^{-t} \leq \|x\|_{\lambda,\infty}(t) \), \( \forall \tau \leq t \)

For an operator \( A : \ell_{\lambda,\infty} \rightarrow \ell_{\lambda,\infty} \), the induced infinity norm is defined as \( \|A\|_{\ell,\infty} = \sup_{x \neq 0, t \geq 0} \|Ax\|_{\ell,\infty} / \|x\|_{\ell,\infty} \).

![Feedback adaptive switching control system \( \Sigma(\hat{K}, P) \)](image)

We consider the system \( \Sigma(\hat{K}, P) \) shown in Fig. 1. It has an unknown plant \( \hat{P} : \ell_{\infty} \rightarrow \ell_{\infty} \), a set \( K \) of \( N \) candidate controllers and the supervisor \( S \). The contents of \( S \) will be explained in subsequent sections. \( \hat{K}(t) \) represents the active controller from \( K \) at time \( t \). \( \xi_P(0) \) refers to the finite initial state of \( P \) at time \( 0 \) and \( \xi_K(t) \) denotes the state of a controller \( K \) at time \( t \). The measured data from the plant \( P \) are denoted by \( \xi = [u \ y]' \).

We consider the special structure [13], [14] for every candidate controller of \( K \). The pair \( (N_i, D_i) \) is the left-coprime matrix fraction description (MFD) of linear \( K_i \in K \) such that \( K_i = D_i^{-1} N_i \) where \( N_i \) and \( D_i \) are exponentially stable (§ III Def. 3) and \( D_i \) is invertible. So \( \xi_{K_i}(t) \) can be expressed as \( \xi_{K_i}(t) = [\xi_{D_i}^{-1}(t) \xi_{N_i}(t)]' \). Let \( K_{i \text{ CLI}} = [D_i N_i] \) and denote by \( \tilde{v}_i \) the fictitious signal (§ III Def. 8)
such that \( \hat{v}_i = K_{CLI}^i \zeta \). The operators \( K_{CLI}^i (i = 1, \ldots, N) \) are called the fictitious signal generators for \( K_i \). We use the notation \( \xi_{K_{CLI}}(t) \) to denote the state of \( K_{CLI}^i \) at time \( t \) and it can be expressed as \( \xi_{K_{CLI}}(t) = [\xi_{D_i}(t), \xi_{N_i}(t)]' \). For a nonlinear \( K_i \in \mathbb{K} \), we have \( \hat{v}_i = -N_i(-y) + D_i u \) where \( N_i \) and \( D_i \) are (possibly nonlinear) incrementally stable (§ III Def. 4) factors and \( D_i \) is invertible.

In this paper, we follow the general approach of [7], [8], but will develop results that relax a number of assumptions and restrictions of the plant and switching algorithm. As in [7], [8], the supervisor uses the hysteresis algorithm (HA) [1] with the additional feature of a reset mechanism, to select a controller from \( \mathbb{K} \) at every instant for \( \Sigma(\hat{K}, P) \). The supervisor evaluates the performance of each candidate controller using a cost function \( V_M \), without necessarily inserting the controllers in the loop. When the plant is time varying then it is important that more importance should be given to the recently evaluated performance than to the old one. To do this, we may select the performance function \( V_M \) so that it depends only on data collected over some recent time-window. One method [9] is to fix the length of time-window for \( V_M \). Another method [7], [8] which we use in this paper, is to use accumulating real-time data to adaptively decide the length of memory time-window. Specifically, we consider an arbitrary partitioning \( Z_+ = \bigcup_{k \in \mathbb{Z}_+} T_k \) where \( T_K = \{t_k, \ldots, t_{k+1} - 1\} \) with \( t_0 = 0 \). Then \( \forall t \in T_K, V_M \) for \( K_i \in \mathbb{K} \) is given by

\[
V_M(K_i, \zeta, t) = \begin{cases} 
\max_{\tau \in [t_{k-M}, t]} V(K_i, \zeta, \tau) & \text{if } t \geq M_k \\
\max_{\tau \in [0, t]} V(K_i, \zeta, \tau) & \text{if } t < M_k
\end{cases}
\]

where \( M_k \in \mathbb{Z}_+ \) is arbitrarily chosen by the designer, and the sequence \( \{t_k\}, k \in \mathbb{Z}_+ \) is a resetting sequence decided adaptively by a reset condition (RC). A RC can be defined as the logic which decides times \( t_k \), \( k \in \mathbb{Z}_+ \) such that the old evaluated performance \( V_M \) prior to \( t_k \) can be safely ignored without inducing instability. Note that, \( V_M \) is monotone in \( t \) over each interval \( T_k \) but not \( \forall t \in Z_+ \).

The function \( V \) is a fading memory cost function which is a positive real valued function. For a time-invariant controller \( K_i \), it is given by

\[
V(K_i, \zeta, t) = \frac{\|u\|_{\ell_{\infty}[0, t]} + \|\hat{v}_i - y\|_{\ell_{\infty}[0, t]}}{\mu} + \|\hat{v}_i\|_{\ell_{\infty}[0, t]}, \quad \mu > 0
\]

The HA orders the candidate controllers based on their evaluated performance \( V_M \) and then switches to the controller with minimum cost. The hysteresis logic (HL) is

\[
\dot{K}(t+1) = \begin{cases} 
\arg \min_{K \in \mathbb{K}} V_M(K, \zeta, t) & \text{if } V_M(\hat{K}(t), \zeta, t) > \min_{K \in \mathbb{K}} V_M(K, \zeta, t) + h_c \\
\hat{K}(t) & \text{otherwise}
\end{cases}
\]

where \( h_c \in \mathbb{R}_+ \) is the hysteresis constant and \( \hat{K}(0) \) is selected from \( \mathbb{K} \) by the designer.

When a controller is switched, undesired transients and discontinuities may occur in the control signal at the switching times. To tackle the undesired bumpy transients associated with switching, we shall adopt the bumpless transfer (BT) method [6], and derive modifications to the BHMT reset condition [7], [8] necessary to preserve convergence and stability with BT.

In summary, our main aim is to improve on the results of [7] by removing the linear plant and linear controller assumptions, and improving the BHMT RC so as to allow BT technique.

### III. Preliminaries

![Control system \( \Gamma(K, P) \) with input \( v \) and output \( \zeta = [u \ y]' \)](image)

**Def. 1:** A function \( \alpha(r) : \mathbb{R} \to \mathbb{R}_+ \) is a class \( K \) function \((\alpha \in \mathcal{K})\) if it is continuous, \( \alpha(0) = 0 \), strictly increasing.

**Def. 2:** A function \( \beta(r, s) : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}_+ \) is a class \( KL \) function \((\beta \in \mathcal{KL})\) if it is continuous, class \( K \) function with respect to \( r \) for each fixed \( s \) and decreasing with respect to \( s \) with \( \lim_{s \to \infty} \beta(r, s) = 0 \) for each fixed \( r \).

**Def. 3:** The system \( \Gamma \) shown in Fig. 2 is said to be exponentially stable with degree \( \lambda_K \) if there exist \( \alpha, \beta \in \mathcal{K} \), constants \( \gamma \in \mathbb{R}_+ \) and \( \lambda_K \in (0, 1) \) such that for each input \( v \in \ell_{\infty} \) and each pair of times \( t \geq \tau \geq 0 \) we have

\[
\|\zeta\|_{\ell_{\infty}[\tau, t]} \leq \alpha(\|v\|_{\ell_{\infty}[\tau, t]} + \beta(\|v\|_{\ell_{\infty}[\tau-1, t-1]}))^{\lambda_K^{-1} + \gamma}
\]

Otherwise \( \Gamma \) is unstable. If (2) holds for \( \lambda = \lambda_K = 1 \) then \( \Gamma \) is said to be stable.

**Def. 4:** The system \( \Gamma \) shown in Fig. 2 is said to be incrementally stable with degree \( \lambda_K \) if there exist \( \alpha, \beta \in \mathcal{K} \), constants \( \gamma \in \mathbb{R}_+ \) and \( \lambda_K \in (0, 1) \) such that for every pair of inputs \( v_1, v_2 \in \ell_{\infty} \) with respective outputs \( \zeta_1, \zeta_2 \) and each pair of times \( t \geq \tau \geq 0 \) we have

\[
\|\zeta_2 - \zeta_1\|_{\ell_{\infty}[\tau, t]} \leq \alpha(\|v_2 - v_1\|_{\ell_{\infty}[\tau, t]} + \beta(\|v_2 - v_1\|_{\ell_{\infty}[\tau-1, t-1]}))^{\lambda_K^{-1} + \gamma}
\]

Otherwise \( \Gamma \) is unstable.

**Def. 5:** The system \( \Gamma \) shown in Fig. 2 is said to be asymptotically stable if there exist \( \alpha \in \mathcal{K}, \beta \in \mathcal{KL} \) and constant \( \gamma \in \mathbb{R}_+ \) such that for each input \( v \in \ell_{\infty} \) and each pair of times \( t \geq \tau \geq 0 \) we have

\[
\|\zeta\|_{\ell_{\infty}[\tau, t]} \leq \alpha(\|v\|_{\ell_{\infty}[\tau, t]} + \beta(\|v\|_{\ell_{\infty}[\tau-1, t-1]}))^{t-\tau+1} + \gamma
\]

Otherwise \( \Gamma \) is unstable.

**Def. 6:** Exponential stability of the system \( \Sigma(\hat{K}, P) \) shown in Fig. 1 is said to be falsified by \( v \) and \( \zeta \) if there exist \( \alpha, \beta \in \mathcal{K} \), constants \( \gamma \in \mathbb{R}_+ \) and \( \lambda_K \in (0, 1) \) such that (2) holds; otherwise it is falsified by \( v \) and \( \zeta \).

According to Def. 6, it can be asserted that the system \( \Sigma(K, P) \) is exponentially stable if and only if its exponential stability is falsified by all possible \( v \) and \( \zeta \). If it can be falsified by \( v \) and \( \zeta \) from even one experiment then \( \Sigma \) is unstable. But if it is falsified by \( v \) and \( \zeta \) from one experiment then it cannot be concluded that \( \Sigma \) is exponentially stable.

**Def. 7:** Given \( \lambda_K \in (0, 1) \), the adaptive stabilization problem for the system \( \Sigma(\hat{K}, P) \) shown in Fig. 1 is said to be \( \lambda_K \)
feasible if there exists at least one exponentially stabilizing controller $K \in \mathcal{K}$ such that (2) holds according to Def. 3.

**Def. 8 (10):** Given the data $\zeta = [u \ y]'$ measured over the period $[t_0, t]$ and a controller $K_i \in \mathcal{K}$, then its fictitious signal $\hat{v}_i$ is a hypothetical input signal that would have exactly reproduced the measured data $\zeta$ had the controller $K_i$ been in the loop for the entire time period $[t_0, t]$ over which the data $\zeta$ were collected.

**Def. 9:** Given a pair $(V, K_i)$, a controller $K_i \in \mathcal{K}$ is said to be $\gamma$ falsified at cost level $\gamma$ at time $t$ by the data $\zeta = [u \ y]'$ measured over $[0, t]$ if $V_{\bar{h}}(K_i, \zeta, t) > \gamma$. Otherwise it is said to be $\gamma$ unfalsified. The real parameter $\gamma$ is called the falsification cost level.

**Def. 10:** The pair $(V, K_i)$ is said to be cost-detectable under the condition that exponential stability of the system $\Sigma(K, P)$ shown in Fig. 1 is unfalsified by $(v, \zeta)$ if and only if $V(K_i, \zeta)$ is a stable signal.

**IV. MAIN RESULTS**

We use the following two assumptions to prove stability results for the system $\Sigma(K, P)$ shown in Fig. 1 under the HA with RC and BT method.

**Assumption A1:** The input signal is bounded i.e. $v \in \ell_{\infty}$.  

**Assumption A2:** We have prior knowledge of a number $\lambda_K \in (0, 1)$ such that $\Sigma(K, P)$ is $\lambda_K$ feasible.

**Lemma 1:** For a real valued signal $x$ defined on $\mathbb{Z}_+$ and for any finite $t \in \mathbb{Z}_+$, the relation between its $\ell_{\infty}$-norms and $\ell_{\infty}$-norms is given by

$$
\|x\|_{\ell_{\infty}[0, t]} \leq \|x\|_{\ell_{\infty}[0, t]} = \sup_{t \in [0, t]} \|x\|_{\ell_{\infty}[0, t]}
$$

**Proof:** Lemma 1 follows directly from the definitions of $\ell_{\infty}$-norms and $\ell_{\infty}$-norms.

For $k^{th}$ reset interval $T_K = \{t_k, ..., t_{k+1} - 1\}, k \in \mathbb{Z}_+$, let

$$
\Pi^k = \{ \min_{K \in \mathcal{K}} \left\{ V(K, \zeta, t) + h_c, t_k \geq M_0 \right\}, \min_{K \in \mathcal{K}} \left\{ V(K, \zeta, t) + h_c, t_k < M_0 \right\} \}
$$

and let $N_k$ denotes the number of switches over $T_k$.

**Lemma 2:** Consider the HA for $\Sigma(K, P)$ with A1, A2. Then there exist positive valued functions $f_1, f_2$ and positive constant $\lambda_K(t)$ such that $\forall t \in T_k$ we have

$$
\|x\|_{\ell_{\infty}[0, t]} \leq f_1(\Pi^k) + f_2(\Pi^k, \lambda_K(t))
$$

**Proof:** Lemma 2 is a consequence of the HL.

Now we prove that there exist an upper bound on the fictitious signal $\hat{v}_i$ of $K_i$ switched on at $t_i$ with finite $\xi_K(t_i)$. $\hat{v}_i$ is generated by $K_{i-1}[L_i]$ with finite $\xi_{K_{i-1}}(0)$.

**Lemma 3:** Consider the HA for $\Sigma(K, P)$ with A1. Suppose $\{K_i\}, i \in \mathbb{Z}_+$ is the switching sequence and $\{t_i\}, i \in \mathbb{Z}_+$ with $t_0 = 0$ is the sequence of switching times. Then for any $K_i$ there exist positive constants $c_1, c_2, c_3, k_0, k_1, ..., k_i$ and $\lambda_i < 1$ associated with $(N_i, D_i)$ such that $\forall t \in [t_i, t_{i+1} - 1]$ we have

$$
\|\hat{v}_i(t) - v\|_{\ell_{\infty}[0, t]} \leq c_1 \|v\|_{\ell_{\infty}[0, t]} + c_2 \|\xi\|_{\ell_{\infty}[0, t]} + c_3 \|\xi_K(t_i)\|_{\ell_{\infty}[t_i, t_{i+1}]} + \max_{j \in [0, t]} \{2k_j\|\xi_{K_j}(t_j)\|_{\ell_{\infty}[t_j, t_{j+1}]}\}
$$

and $\lim_{t \to \infty, t_i \to \infty} |\hat{v}_i(t) - v(t)| = 0$

**Proof:** Refer the appendix.

In [10], [7], it is proved that the fictitious reference signal $\hat{v}$ of the active controller converges to the actual $v$ under A2, and only for the finite state determined by the closed loop system for the switched controller at its switching time. But Lemma 3 implies that $\hat{v}$ of the active controller converges to $v$ without A2, and the result is applicable for any finite state of the switched controller at its switching time. Theoretically the result holds even for a destabilizing controller but practically it can be verified by normalizing and scaling unbounded signals $u$ and $y$ which are inputs to a generator $K_{CL}$.

The next lemma shows that for every $T_k$ there exists an upper bound on $\zeta$ and its dependency on how we initialize states of controllers in the switching sequence at their respective switching times.

**Lemma 4:** Consider the HA for $\Sigma(K, P)$ with A1, A2. Then there exist positive valued functions $f_1, f_2$ and positive constant $\lambda_K(t)$ such that $\forall t \in T_k$ we have

$$
\|x\|_{\ell_{\infty}[0, t]} \leq f_1(\Pi^k) + f_2(\Pi^k, \lambda_K(t))
$$

where

$$
f_1(\Pi^k) = \sum_{n=0}^{N_k} \left[ h(\Pi^k) \right]^n, f_2(\Pi^k, \lambda_K(t)) = \sum_{n=0}^{N_k} \left[ \max_{j \in [0, N_n]} \left\{ \|\xi_{K_j}(t_{j+k})\|_{\ell_{\infty}[t_{j+k}, t_{j+k}]} \right\} \right]
$$

and $\{t_{k[1]}, t_{k[2]}, ..., t_{k[N_n]}\}$ is the sequence of controller switching times over $T_k$ and $h(\Pi^k)$ is given in (17) of the appendix.

**Proof:** Refer the appendix.

**A. Bumpless Transfer Technique**

We use the BT method [6] with slight modification to remove the abrupt changes in $u$ due to a controller output signal mismatch at the switching times. Suppose the controller $K_i$ is switched on at $t_i$. Consider the slow-fast controller decomposition for $K_i$ as $K_i = K_{i-slow} + K_{i-fast}$ with the minimal realizations

$$
K_{i-slow} = \left[ \begin{array}{c|c} A_{fs} & B_{fs} \\ \hline C_{fs} & D_{fs} \end{array} \right], K_{i-fast} = \left[ \begin{array}{c|c} A_{fs} & B_{fs} \\ \hline C_{fs} & D_{fs} \end{array} \right]
$$

where $K_{i-slow}$ is in the observable canonical form. We want $u(t_i)$ to be as if there is no switch provided $y(t_i) - y(t_i - 1) = y(t_i - 1) - y(t_i - 2)$ and $v(t_i) - v(t_i - 1) = v(t_i - 1) - v(t_i - 2)$. Therefore by the linear interpolation, we predict the input to $K_i$ at $t_i$ and then decide $\xi_{K_i}(t_i)$.
such that \( u(t_i) = u(t_i - 1) \). So re-initialize the state \( \xi_{K_i}(t_i) = [\xi_{K_i - \text{fast}}(t_i) \xi_{K_i - \text{slow}}(t_i)]' \) as \( \xi_{K_i - \text{fast}}(t_i) = \ldots + f_2(\Pi_k)\bar{\lambda}_{tk+1-tk} + f_2(\Pi_k)\bar{\lambda}_{tk+1-tk} f_2(\Pi_{k-1}) \bar{\lambda}_{tk-tk-1} + \ldots + f_2(\Pi_k)\bar{\lambda}_{tk+1-tk} f_2(\Pi_0)\bar{\lambda}_{t1} \) for each controller in the switching sequence at their respective switching times in the following way.

**Theorem 1:** Consider the HA with BT (3) for \( \Sigma(\hat{K}, P) \) with assumptions made in Lemma 3. Then for any \( K_i, i \in \mathbb{Z}_+ \) there exist positive constants \( \alpha, \beta, \gamma \) and \( \lambda_1 < 1 \) associated with \( (N_i, D_i) \) such that \( \forall t \in [t_i, t_{i+1} - 1] \) we have

\[
\| \hat{v}_i - v \|_{\infty, [0,t]} \leq \left[ \alpha \| v \|_{\infty, [0,t-1]} + \beta \| \zeta_i \|_{\infty, [0,t-1]} + \gamma \| \Pi_{K_i-\text{fast}}(t) \|_{\infty} \right] \bar{\lambda}_{t-i+1}^{-1}
\]

**Proof:** Refer the appendix.

**Theorem 2:** Consider the HA with BT (3) for \( \Sigma(\hat{K}, P) \) with A1, A2. Then there exist positive valued functions \( f_1, f_2, f_3 \) and positive constant \( \lambda_{K_i}(t) \) such that \( \forall t \in T_k \) we have

\[
\| \zeta \|_{\infty, [0,t]} \leq f_1(\Pi^k) \left[ \mu + \| \zeta P(0) \|_{\infty} + f_3(\Pi^k) \right] + f_2(\Pi^k) \| \zeta \|_{\infty, [0,t-1]} \lambda_{K_i}^{-1} + f_3(\Pi^k) \| v \|_{\infty, [0,t]} \]

(4)

where \( f_1(\Pi^k) = \sum_{i=0}^{N_k} \left[ 1 + M_i^* \right]^{i} \| h(\Pi^k) \|_{\infty} \lambda_{K_i}^{i+1} \)

\[
f_2(\Pi^k) = \left[ 1 + M_1^* \right] h(\Pi^k)^{N_k+1} \lambda_{K_i}^{-1} + M_1^* \sum_{i=0}^{N_k} \left[ 1 + M_i^* \right]^{i} \| h(\Pi^k) \|_{\infty} \lambda_{K_i}^{i+1}
\]

with \( M_1^* = \max_{i \in N} \| M_1 \|_{\infty} \lambda_{K_i} \).

**Proof:** Refer the appendix.

**B. Reset Condition**

From (4) it follows that, if for some \( \rho < 0, 1 \) it holds \( \forall t \in T_k \) that \( f_2(\Pi^k) \lambda_{K_i}^{-1} \leq \rho \), then \( \forall t \in T_k \) we have

\[
\| \zeta \|_{\infty, [0,t]} \leq f_1(\Pi^k) \left[ \mu + \| \zeta P(0) \|_{\infty} + f_3(\Pi^k) \right] \| v \|_{\infty, [0,t]} \rho^{-1}
\]

Therefore the resetting sequence \( \{ t_k \} \) should be chosen in such a way that \( f_2(\Pi^k) \lambda_{K_i}^{-1} \leq \rho \) at the end of each \( T_k \); i.e., \( f_2(\Pi^k) \lambda_{K_i}^{-1} \leq \rho < 1, \forall k \in \mathbb{Z}_+ \).

Consider the \( n^{th} \) sub-interval \( T_{k[n]} = \{ t_{k[n]}, \ldots, t_{k[n+1]} - 1 \} \) of \( T_k \) with \( t_{k[0]} = t_k \) such that the switching signal is constant over \( T_{k[n]} \). Assume that when \( t \in T_{k[n]} \), \( n \) switches have happened in \( T_k \). We want to decide \( t_{k[n+1]} > t_k[n] \) such that \( f_2(\Pi^k) \lambda_{K_i}^{-1} \leq \rho \) where \( f_2(\Pi^k) = [1 + M^*_i] h(\Pi^k)^{n+1} \).

Remark: If \( K_i \in \mathbb{K} \) does not have the slow modes then its state space realization can be augmented with the additional stable but uncontrollable slow modes that were not originally contained in \( K_i \). This controller augmentation technique also works for nonlinear controllers.

Now we can extend the results of Lemmas 3 and 4 for the BT method by considering the gains \( (M_1, M_2) \) established in (3) for each controller in the switching sequence at their respective switching times in the following way.

By Lemma 2, there exists \( N_\rho \in \mathbb{Z}_+ \) such that \( N_\rho \geq N_k, \forall k \in \mathbb{Z}_+ \). By A2, \( \Pi \) is finite for each \( K_i \in \mathbb{K} \) which implies that there exists \( \Pi^* \in \mathbb{R}^+ \) such that \( \Pi^* \geq \Pi \), \( \forall k \in \mathbb{Z}_+ \). Now we prove that the stability of \( \Sigma(\hat{K}, P) \) is still preserved even if the sequence of reset intervals is infinite.

**Theorem 3 (Main Result):** Consider the HA with BT (3) and RC (5) and BT (3) for \( \Sigma(\hat{K}, P) \) with A1, A2. Then the system \( \Sigma(\hat{K}, P) \) is stable and moreover there exist positive valued functions \( f_1, f_3 \), constant \( \epsilon > 0 \) such that \( \forall t \in \mathbb{R}_+ \) we have

\[
\| \zeta \|_{\infty, [0,t]} \leq f_1(\Pi^*) \epsilon [ \mu + \| \zeta P(0) \|_{\infty} + f_3(\Pi^*) \epsilon ] \| v \|_{\infty, [0,t]} \rho^{-1}
\]

where \( f_1(\Pi^*), f_3(\Pi^*) \) are as in Theorem 2 using \( N_\rho \).

**Proof:** By the RC (5) used in Algorithm and a cost-detectable pair of \( V \) and feasible controller under A2, we have

\[
f_2(\Pi^k) \lambda_{K_i}^{-1} \leq \rho, \forall k \in \mathbb{Z}_+ \text{ where } \rho < 0.
\]

Therefore

\[
\lim_{t \to \infty} \left[ 1 + f_2(\Pi^k) \lambda_{K_i}^{-1} t_k + f_2(\Pi^k) \lambda_{K_i}^{-1} t_k f_2(\Pi^k) \right] \lambda_{K_i}^{-1} \leq \ldots \leq f_2(\Pi^k) \lambda_{K_i}^{-1} t_k + f_2(\Pi^k) \lambda_{K_i}^{-1} t_k f_2(\Pi^k) \lambda_{K_i}^{-1} \leq \ldots \leq f_2(\Pi^k) \lambda_{K_i}^{-1} t_k = \frac{1}{1-\rho}
\]
Let $\epsilon = 1/(1 - \rho)$. We have $N_k \geq N^*$ and $\Pi^* \geq \Pi^k, \forall k \in \mathbb{Z}_+$. Therefore $f_1(\Pi^k) < f_1(\Pi^*)$ and $f_3(\Pi^k) < f_3(\Pi^*), \forall k \in \mathbb{Z}_+$ using $N_k$. Hence by the principle of induction, using (4) over each reset interval from $T_0$ to $T_k$ and taking limit as $k \to \infty$ with $\|v\|_{\infty[0,t]} \leq \|v\|_{\infty[0,t]}$ (by Lemma 1), $\forall t \in \mathbb{Z}_+$ we have

$$\|\xi\|_{\infty[0,t]} \leq f_1(\Pi^*)\epsilon [\mu + \|\xi_P(0)\|_{\infty}] + f_3(\Pi^*)\epsilon [\|v\|_{\infty[0,t]} = \Delta(\Pi^*) (6)$$

Since (6) is the ultimate upper bound on $\xi$, using Lemma 1 $\|\xi\|_{\infty[0,t]} = \sup_{r \in [0,t]} \|\xi\|_{\infty[0,r]} \leq \Delta(\Pi^*)$.

C. Time-Varying Plants

The following lemmas provide an expression for the upper bound $\Pi^*$ on the falsification cost for the situation in which the plant time variation consists of a sequence of possibly time-varying frozen-time plants, each of which satisfies the feasibility condition.

**Lemma 5:** Consider the HA with RC and BT (3) for $\Sigma(\hat{K}, P)$ with A1, A2. For the given time-varying plant $P$, consider its frozen-time plant $P_\tau$ over $\{\tau, ..., T\}$ such that under A2 there is at least one exponentially stabilizing controller $K_1 \in \mathbb{K}, \forall t \in [\tau, T]$. Then there exist positive constants $k_1, ..., k_4$ and $\lambda_{K(t)}$ such that $\forall t \in [\tau, T]$ we have

$$V(K_i, \xi, t) \leq k_1 + k_2 \|v\|_{\infty} + k_3 \|\xi\|_{\infty} + k_4 \|\xi_P(0)\|_{\infty} + \epsilon \Pi^*$$

**Proof:** Refer the appendix.

**Corollary 1:** In Lemma 5, if there exists a single frozen time plant $P_\tau$ for all $t \in \mathbb{Z}_+$ under A2, then there exists some $K_1 \in \mathbb{K}$ such that $\forall t \in \mathbb{Z}_+$ we have

$$V(K_i, \xi, t) \leq k_1 + k_2 \|v\|_{\infty} + k_4 \|\xi_P(0)\|_{\infty} + \epsilon \Pi^*$$

For the given time-varying plant $P$, consider a sequence of its frozen time plants $\{P_{F_m}, P_{F_1}, ...\}$ such that under A2 for each $P_{F_m}$ there exists at least one exponentially stabilizing controller and there may or may not exist one single exponentially stabilizing controller $\forall P_{F_m}, m \in \mathbb{Z}_+$. Let $P_{F_m}$ exists for period $T_{F_m} = [t_m, t_{m+1} - 1]$ where $m \in \mathbb{Z}_+$. By Lemma 5 and Corollary 1, for every $m$ there exists some $K_i \in \mathbb{K}$ such that $\forall t \in T_{F_m}$ we have $V(K_i, \xi, t) \leq \Pi^* + k_3 \|\xi\|_{\infty} + k_4 \|\xi_P(0)\|_{\infty} + \epsilon \Pi^*$. Given $\eta > 0$, let $T_{F_m} = \{t \in T_{F_m} | k_3 \|\xi\|_{\infty} + k_4 \|\xi_P(0)\|_{\infty} + \epsilon \Pi^* < \eta\}$ which implies $V(K_i, \xi, t) \leq \Pi^* + \eta, \forall t \in T_{F_m}$. Therefore by (6), $\|\xi\|_{\infty} < \Pi^* + \eta, \forall t \in T_{F_m}$ which implies $V(K_i, \xi, t) \leq \Pi^* + \eta, \forall t \in T_{F_m}$ by property (b). Similarly considering $T_{F_m}$, we have $\|\xi\|_{\infty} < \Pi^* + \eta$ which implies $V(K_i, \xi, t) \leq \Pi^* + \eta, \forall t \in T_{F_m}$ - $T_{F_m}$. Using these arguments we mention the next lemma.

**Lemma 6:** Consider the HA with RC and BT (3) for $\Sigma(\hat{K}, P)$ with A1, A2. For the given $\eta > 0$ and time-varying plant $P$, consider its frozen time plant sequence $\{P_{F_m}\}$ such that $\forall m \in \mathbb{Z}_+, \exists k \in \mathbb{Z}_+$ such that $T_k \in T_{F_m}$. Then there exists constant $k > 0$ such that $\forall k \in \mathbb{Z}_+$ we have

$$\Pi^k \leq \Pi^* + k\Delta(\Pi^* + \eta) = \Pi^*$$

where $\Delta(\cdot)$ is as in (6).

Lemma 6 indicates stability holds for the time-varying $P$ under the additional constraint ($\eta$) that $P$ should vary infrequently. Small $\eta$ implies variation in $P$ should occur at large intervals. Large $\eta$ implies $P$ can vary at any rate, but $\mathbb{K}$ should always have at least one exponentially stabilizing controller which can stabilize $P$ in small time. So Lemma 6 confirms the obvious idea of slowness of time-varying plant.

Therefore by Theorem 3 and Def. 3, the system $\Sigma(\hat{K}, P)$ shown in Fig. 1 with A1, A2 is stable $\forall t \in \mathbb{Z}_+$. It is also exponentially stable over each $T_k$ by Theorem 2 and Def. 3.

To decide a resetting instant by the RC (5), we wait until $f_2(\cdot)\lambda^{t-h_k}$ decreases exponentially below $\rho$. But it can reduce asymptotically to $P$ and stay below it without converging to zero. Under this condition, by relaxing exponentially decaying term $\lambda_{K(t)}$ in Theorem 2 and by Def. 5, the system $\Sigma(\hat{K}, P)$ is asymptotically stable over each $T_k$.

V. SIMULATION EXAMPLE

Consider the linear time-varying unstable plant $P = (s^2 + s + 1)/(s^3 + s^2 + 98s - 100)$ with two PID controllers $K_1 = 80 + 50/s + 0.5s/(0.1s + 1)$ and $K_2 = 5 + 2/s + 1.25s/(0.1s + 1)$. Fig. 3 shows the closed loop step behavior of discretized $P$ with $K_1, K_2$ under two cases (i) HA with RC (5) and (ii) HA with RC (5) and BT (3). The plant $P$ changes to $\hat{P} = (s + 10)/(s^2 + 98s)$ at $t = 7000$. $K_1$ stabilizes $P$ while $K_2$ cannot. But $K_1, K_2$ both stabilize $\hat{P}$ with $K_1$ providing best performance. $P, K_1, K_2, K_{cli}^2$ and $K_{cli}^3$ are initialized with zero initial conditions. Assume $h_e = 0.01, \mu = 3, \lambda = \rho = 0.95, M_s = 50$. We can observe that falsification cost didn’t increase continuously and BT (3) removed the undesired bumpiness. Even for the changed plant, it selected the best performing controller $K_1$.

VI. CONCLUSION

We looked at the problem of improving the unfalsified controller BHMT reset condition of [7], [8] by removing
the linear plant and the linear controller assumptions, and modifying the reset condition as needed to accommodate humpless transfer. Our main result (Theorem 3) guarantees stability and convergence, subject only to feasibility of the adaptive control problem. Further, the results of Lemmas 5 and 6 also imply stability for the case of infrequently time-varying plants.

APPENDIX

With reference to (3), we will need constants $M_1 = \max_{i \in N} \| M_1 \|_{\infty}$, $M_2 = \max_{i \in N} \| M_2 \|_{\infty}$ in proofs. Note that, by A1 and Lemma 1 $v \in \mathcal{E}_{\infty}$.

**Lemma 3 Proof:** Let $\{ K_0, K_1, \ldots \}$ be the switching sequence with $(t_0 = 0), t_1, \ldots$ be the sequence of switching times. Consider any $i$th active controller $K_{i}$ from $\{ K_0, K_1, \ldots \}$ over the interval $[0, t_{i+1} - 1]$ where $i \in \mathbb{Z}_+$. Let the pair $(N_i, D_i)$ be the left-coprime MFD of $K_i$. Let $\hat{\xi}_{K_i}^{(\oe)}(t_i) = \xi_{K_i}^{(\oe)}(t_i)$ be the state at time $t_i$ as shown in Fig. 4. Assume $K_{CL}^{i}$ has finite $\xi_{K_{CL}^{i}}^{(\oe)}(0)$. Its state at time $t_i$ is $\xi_{K_{CL}^{i}}^{(\oe)}(t_i) = [\xi_{K_{CL}^{i}}^{(\oe)}(t_i) \xi_{N_i}^{(\oe)}(t_i)]^T$ which is unchanged as shown in Fig. 4. Since $\hat{v}_i = N_i y + D_i u$, let $u = D_i^{-1} (v - N_i y)$ then $N_i$ and $D_i$ are exponentially stable, there exist positive constants $k_i$ and $\lambda_i < 1$ associated with $(N_i, D_i)$ such that $\forall t \in [t_i, t_{i+1} - 1]$ we have $\| \hat{v}_i - v \|_{\lambda_0[t_i, t]} \leq [k_i \| \xi_{K_{CL}^{i}}^{(\oe)}(t_i) \|_{\infty}]^{t_{i+1}}$.

Since $K_{CL}^{i}$ is an exponentially stable generator, there exist positive constants $a_1$ and $a_2$ such that $\| \hat{v}_i - v \|_{\lambda_0[t_i, t]} \leq [a_1 \| \xi_{K_{CL}^{i}}^{(\oe)}(t_i) \|_{\infty}]^{t_{i+1}} + k_i \| \xi_{K_i}^{(\oe)}(t_i) \|_{\infty}]^{t_{i+1}}$ (7)

Using property (a), $\forall t \in [t_i, t_{i+1} - 1]$ we have $\| \hat{v}_i - v \|_{\lambda_0[t_i, t]} \leq [\| \hat{v}_i - v \|_{\lambda_0[t_i, t]} + \| \hat{v}_i - v \|_{\lambda_0[t_i, t] - 1}]^{t_{i+1}}$ (8)

![Fig. 5. Generator of $\hat{v}_i$ over $[0, t_i - 1]$.](image)

![Fig. 4. Generator of $\hat{v}_i$ over $[t_i, t_{i+1} - 1]$.](image)

Theorem 3 is derived from Theorem 2. For the system $\Sigma(K, P)$ shown in Fig. 1 with A1, A2, there exist constants $k_1, k_2, k_3 \in \mathbb{R}_+$ such that for each pair of times $t \geq 0$ $\| v \|_{\lambda_0[t, t]} \leq k_1 \| v \|_{\lambda_0[t, t]} + k_2 \| [\xi_{K_i}^{(\oe)}(t_i)] \|_{\infty}]^{t_{i+1}}$  $+ k_3 \| \xi_{P}(0) \|_{\infty}]^{t_{i+1}}$ (15)

An upper bound on $\| v \|_{\lambda_0[t, t]}$ is derived from Theorem 2 because at $t'$ active controller $K_{i}$ is falsified, so we cannot prove that $V(K_{i}, \xi, t') \leq \Pi_k$. For the system $\Sigma(K, P)$ shown in Fig. 1 with A1, A2, there exist constants $k_1, k_2, k_3 \in \mathbb{R}_+$ such that for each pair of times $t \geq 0$ $\| v \|_{\lambda_0[t, t]} \leq k_1 \| v \|_{\lambda_0[t, t]} + k_2 \| [\xi_{K_i}^{(\oe)}(t_i)] \|_{\infty}]^{t_{i+1}}$  $+ k_3 \| \xi_{P}(0) \|_{\infty}]^{t_{i+1}}$ (15)

Hence with $t = \tau' = \tau$ in (15), we get an upper bound on $\| v \|_{\lambda_0[0, \tau]}$ because $\tau' - 1 \in T_{k[i]} - \{ \tau \}$. But by (14) there exists an upper bound on $\| v \|_{\lambda_0[0, \tau - 1]}$ because $\tau' - 1 \in T_{k[i]} - \{ \tau \}$.
An upper bound on $\|\xi\|_{\lambda,0,[t_k]}$: Consider a constant $k \in \mathbb{R}_+$ such that $k > k_j, \forall j \in \mathbb{N}$. Let $h_1(\Pi_k) = \Pi_k, h_2(\Pi_k) = c_1(\Pi_k^2) + 1$ and $h_3(\Pi_k) = 2k(\Pi_k^2 + 1)$. Let $h(\Pi_k)$ be a non-negative function such that $h(\Pi_k) \geq h_i(\Pi_k), i = 1, 2, 3, 4$. It is given by

$$h(\Pi_k) = 1 + 2c(\Pi_k^2 + 1)$$

where $c = \{c_1, c_2, k\}$.

Therefore by (14) and (16), $\forall t \in T_{k[i]}$ we have

$$\|\xi\|_{\lambda,0,[t_k]} \leq h(\Pi_k) \left[\mu + \|v\|_{\lambda,0,[t]} + \|\xi_P(0)\|_{\lambda,\hat{K}_i} + \max_{j \in [0,k], j \in \{0,N_i\}} \left\{\|\xi_{K_j,i}(\xi_{K_j,i})\|_{\lambda,\hat{K}_i} + \max_{j \in [0,N_i]} \left\{\|\xi_{K_j,i}(t_{k[j]}(t))\|_{\lambda,\hat{K}_i} \right\}\right\} \right]

By Lemma 2, the number of switches $N_k$ over $T_k$ is finite. So there must be at most $N_k + 1$ finite number of sub-intervals over $T_k$. Therefore the claim follows by the principle of induction using (18) over each sub-interval of $T_k$.

**Theorem 1 Proof:** Consider the BT (3) method to re-initialize the states of all switched controllers at their respective switching times. Therefore by using (3) in Lemma 3 with $k^* = \max_{i \in \mathbb{N}} k_i$, $\forall t \in [t_k, t_{k+1}]$ we have

$$\|\bar{v}_i - v\|_{\lambda,0,[t_k]} \leq \left[\|v\|_{\lambda,0,[t_k]} + c_2\|\xi\|_{\lambda,0,[t_k]} \right]

+ c_3(\xi_{K,i}(0))_{\lambda,\hat{K}_i} + \max_{j \in [0,N_i]} \left\{\|M_{2,k}\|_{\lambda,\hat{K}_i} \right\}

\|\bar{v}\|_{\lambda,0,[t_k]} 

\leq \left[\|v\|_{\lambda,0,[t_k]} + c_2\|\xi\|_{\lambda,0,[t_k]} + c_3(\xi_{K,i}(0))_{\lambda,\hat{K}_i} + \max_{j \in [0,N_i]} \left\{\|M_{2,k}\|_{\lambda,\hat{K}_i} \right\} \right]

So the claim follows by letting $\alpha = c_1 + 2k^*M^2, \beta = c_2 + 2k^*M^2$ and $\gamma = c_3$.

**Theorem 2 Proof:** Consider the $i$th sub-interval $T_{k[i]}$ of $T_k$ over which $\hat{K}_i$ is active. Consider the BT (3) method to re-initialize the states of all switched controllers at their respective switching times. Therefore by using (3) in Lemma 3, $\forall t \in T_{k[i]}$ we have

$$\|\xi\|_{\lambda,0,[t_k]} \leq h(\Pi_k) \left[\mu + \|v\|_{\lambda,0,[t_k]} + \|\xi\|_{\lambda,0,[t_k]} + \|\xi_P(0)\|_{\lambda,\hat{K}_i} + \max_{j \in [0,k], j \in \{0,N_i\}} \left\{\|\xi_{K_j,i}(\xi_{K_j,i})\|_{\lambda,\hat{K}_i} + \max_{j \in [0,N_i]} \left\{\|\xi_{K_j,i}(t_{k[j]}(t))\|_{\lambda,\hat{K}_i} \right\}\right\} \right]

\|\xi\|_{\lambda,0,[t_k]} \leq \left[\|v\|_{\lambda,0,[t_k]} + c_2\|\xi\|_{\lambda,0,[t_k]} + c_3(\xi_{K,i}(0))_{\lambda,\hat{K}_i} + \max_{j \in [0,N_i]} \left\{\|M_{2,k}\|_{\lambda,\hat{K}_i} \right\} \right]

\|\xi\|_{\lambda,0,[t_k]} \leq \left[\|v\|_{\lambda,0,[t_k]} + c_2\|\xi\|_{\lambda,0,[t_k]} + c_3(\xi_{K,i}(0))_{\lambda,\hat{K}_i} + \max_{j \in [0,N_i]} \left\{\|M_{2,k}\|_{\lambda,\hat{K}_i} \right\} \right]

So the claim follows by (20), (22) and (23) with $k_{1i} = c_1(b_1 + c_0)/\mu, k_{2i} = (c_1 + c_2)/\mu, k_{3i} = c_3(b_i + c_0)/\mu$.

**References**


