The Round-Trip Contrast Problem

John Marriott

Abstract—The contrast problem in nuclear magnetic resonance imaging has been treated by a variety of recent works. In this process, the image is typically captured multiple times to filter noise from the image, and to do so the system must return to its initial state before repeating the experiment. A natural motivation is to perform this return trip as quickly as possible, in particular with live subjects.

In this article we introduce and discuss preliminary results of this so-called return trip of the contrast problem. The tools of geometric optimal control theory have been effectively applied to the contrast problem, and they are similarly employed to this problem which shares many characteristics. The time-minimal transfer in the single-spin case and preliminary results in the two-spin case are presented.

I. INTRODUCTION

A number of recent works have examined the so-called contrast problem in nuclear magnetic resonance (NMR) from a variety of angles; numerical algorithms for construction of controls [1], [2], [3], experimental implementation [4], [5], and analysis of the geometric properties of the control system [6], [7], [8]. Here we address a second component of the overall imaging process, the return of the system to its initial configuration.

A. Physical model

The magnetization of a nucleus, which corresponds to its quantum spin, is represented by a tuple \( q_i = (x_i, y_i, z_i)^T \) belonging to the Bloch ball, \( B_i := \{ q_i : |q_i| \leq 1 \} \). Its dynamics are given by the Bloch equation [9]

\[
\begin{align*}
\dot{x}_i &= -\gamma_i x_i + u_y z_i \\
\dot{y}_i &= -\gamma_i y_i - u_z z_i \\
\dot{z}_i &= \gamma (1 - z_i) + (u_x y_i - u_y x_i).
\end{align*}
\]

where \((\gamma_i, \Gamma_i)\) is a parameter pair for a given substance corresponding to the \( T_1 \) and \( T_2 \) relaxation times of the nucleus \((\gamma = (T_1 \cdot 32.3)^{-1} \text{ and } \Gamma = (T_2 \cdot 32.3)^{-1})\), which are subject to the physical constraint \( 2\Gamma_i \geq \gamma_i \geq 0 \) [10] for which \( B_i \) is invariant for the dynamics. Here, \( u = (u_x, u_y) \) represents an external magnetic field, the means of control, which is bounded. We take \(|u| \leq 1\).

For the uncontrolled system, \( u = 0 \), the north pole \( N_i := (0,0,1) \) is a globally attractive equilibrium point, and so is taken as the initial state. We additionally denote \( O_i := (0,0,0) \).

We will consider two such magnetizations, denoted \( q_1 \) and \( q_2 \) with respective parameters \((\gamma_1, \Gamma_1)\) and \((\gamma_2, \Gamma_2)\), both subject to (1). We define \( N = N_1 \times N_2 \).

We will use the parameters of deoxygenated and oxygenated blood for illustration. In the normalized coordinates, these values are \( \gamma_1 = 1/(1.35 \cdot 32.3) \) and \( \Gamma_1 = 1/(0.05 \cdot 32.3) \) for deoxygenated blood and \( \gamma_2 = 1/(1.35 \cdot 32.3) \) and \( \Gamma_2 = 1/(0.2 \cdot 32.3) \) for oxygenated blood.

B. Problem statement

In NMR imaging the magnitude of the magnetization, \(|q|\), assigns the nucleus its color, from \(|q| = 0\) as black to \(|q| = 1\) as white on a gray scale.

In the contrast problem, a control is applied to maximize the visual contrast between two substances in a heterogeneous mixture, i.e., to separate as much as possible the magnitudes of their magnetization (this difference being the visual contrast in the image).

A special case is the contrast problem by saturation in which the first magnetization is set to zero, \( q_1(T) = 0 \), which can be fixed or free, the final state \( q_1(T) = O_1 \) while minimizing \( J(u) = -||q_2(T)||^2 \), with \( q_1 \) and \( q_2 \) subject to the dynamics (1) where \( u \) is measurable and \(|u| \leq 1\).

With this cost functional, this is a Mayer problem. The previously-referenced works consider this formulation of the problem, and among these results there are strong indications that a global optimum is approached [2].

In practice the image is obtained multiple times to filter noise from data obtained in the imaging process. The heuristic approach is to apply the contrast-producing control, then simply allow the uncontrolled system to relax to equilibrium before repeating the experiment. For typical physical parameters, this heuristic return trip is roughly 10 times the time duration of the contrast-producing step. It is desirable to reduce the duration of this step, particularly in NMR imaging of live subjects.

We therefore seek to minimize the time taken by this return trip. Thus our objective is to take the outcome of the contrast-producing control, a state of the form \( O_1 \times q_2 \) as the initial point and to return the state to its pre-contrast configuration in minimum time. Although the point \( q_2 \) is in some sense known, we consider it to be arbitrary. For reasons that will be discussed below, we additionally consider an arbitrary terminal point \( N' \) instead of \( N \) (in particular we take \( N' \) on the steady-state ellipse, defined below). The problem is then:

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J. Marriott is with the Department of Mathematics, University of Hawai‘i at Mānoa, Honolulu, HI 96822 USA

marriott@math.hawaii.edu
The return-trip problem: From an initial point \(O_1 \times q_2\) reach in minimum time \(T\) the final state \(q_1(T) \times q_2(T) = N_0\) with \(q_1\) and \(q_2\) subject to the dynamics (1) where \(u(\cdot)\) is measurable and \(||u|| \leq 1\).

C. The maximum principle

We recall Pontryagin’s maximum principle [11].

Proposition 1: Let the state dynamics be \(\dot{x} = F(x,u)\), \(N(x)\) be a function defining the terminal set, \(c\) be the cost functional, and the control be bounded by \(||u|| \leq M\). If \(u^*\) with corresponding trajectory \(x^*\) is optimal then the following necessary optimality conditions are satisfied.

Denoting \(H(x,p,u) = \langle p, F(x,u) \rangle\) as the pseudo-Hamiltonian, there exists \(p^*(\cdot)\) and a constant \(p_0 \leq 0\) such that for almost every \(t \in [0,T]\),

(i) \(\frac{dx}{dt} = \frac{\partial H}{\partial p}(x^*,p^*,u^*)\), \(\frac{dp}{dt} = -\frac{\partial H}{\partial x}(x^*,p^*,u^*)\)

(ii) \(H(x^*,p^*,u) = \max_{s \in U} H(x^*,p^*,v)\) (maximization condition)

(iii) \(p^*(t), p_0 \neq (0,0)\) (nontriviality condition)

(iv) \(p^*(T) = p_0 = \frac{\partial H}{\partial x}(x^*(T)) + \sum \sigma_i \frac{\partial H}{\partial x}(x^*(T))\), \(\sigma = (\sigma_1, \ldots, \sigma_k) \in \mathbb{R}^k\) (transversality condition)

A pair \((x, p)\) which satisfies these conditions is called an extremal. By the maximum principle, extremals are necessarily concatenations of bang and singular arcs.

In the single-input case with dynamics \(\dot{x} = A + uB\), for the problem types considered here (time-minimization and Mayer-type problems) we have the following concerning the structure of optimal controls. By the maximization condition of Prop. 1, the control value is determined by \(u = \text{sgn}(p,B)\) if this inner product is nonzero. Times at which this inner product changes sign correspond to switchings of the control value between \(\pm 1\), and if the number of switchings is finite we say that the extremal is bang-bang on this interval. If \(\langle p, B \rangle\) is identically zero on an interval, this is the so-called singular case, and the singular control value is computed by relations on the derivatives of \(H(x,p,u)\) since they are identically zero as well.

These derivatives are neatly captured by the Lie bracket: we have that for a vector field \(X\), \(\frac{d}{dt} \langle p, X \rangle = \langle p, [A,X] + u[B,X] \rangle\), where we use the convention that \([X,Y](x) = \frac{\partial Y}{\partial x}(x) \cdot X(x)\). Using this rule, we have that \(\frac{d}{dt} \langle p, B \rangle = \langle p, [A,B] \rangle\) and \(\frac{d^2}{dt^2} \langle p, B \rangle = \langle p, [A,[A,B]] \rangle + u \langle p, [B,[A,B]] \rangle\). If \(\langle p, [B,[A,B]] \rangle\) is nonzero these relations define the order one singular control value

\[
  u_s = -\frac{\langle p, [A,B] \rangle}{\langle p, [B,[A,B]] \rangle}
\]

defined on the singular manifold

\[
  \langle p, B \rangle = \langle p, [A,B] \rangle = 0 \quad \text{and} \quad \langle p, [B,[A,B]] \rangle \neq 0.
\]

In order to classify the time-optimality of a singular arc, we use the strengthened Legendre–Clebsch condition, that

\[
  \langle p, [B,[A,B]] \rangle > 0
\]

along an optimal solution [12].

II. THE CONTRAST PROBLEM

Here we summarize results concerning the contrast problem that are relevant in the return-trip problem.

Observe that the so-called saturation problem plays a key role in the analysis of the contrast problem. When a magnetization is at the origin it is said to be saturated. The saturation problem is to control a single spin from \(N_1\) to \(O_1\) in minimum time. This is relevant in the contrast problem for several reasons. First, it serves as a limit case where the final time is fixed to the minimum time needed for the first spin to reach the origin. Second, it can be shown that a solution of the saturation problem can be embedded as an abnormal extremal solution of the contrast problem [6]. Finally, it serves as an initial extremal for numerical continuation methods [1]. Thus we recall the results of the single-spin saturation problem [13], [14] and their application to the contrast problem.

A. The single-spin case: geometry

Here we consider a single spin \(q = (x,y,z)\) with parameters \((\gamma, \Gamma)\), governed by (1). We also denote \(\delta := \gamma - \Gamma\). Note that due to the symmetry of revolution of the system about the \(z\)-axis [6], without loss of generality we can set \(u = 0\) so that \(x = 0\) which reduces the problem to a two-dimensional state and single-input control.

In this case we write \(u = u_s\) and the dynamics (1) as

\[
  \dot{y} = -\delta y - uz, \quad \dot{z} = \gamma(1-z) + uy
\]

or compactly as \(\dot{q} = F(q) + uG(q)\) with \(F := (-\Gamma y, \gamma(1-z))^T\) and \(G := (-\delta, y)^T\).

One object of interest is the collinearity set, the set of points such that \(F\) and \(G\) are collinear, or equivalently that \(\det(F,G) = 0\). By calculation, this is given by the relation \(\Gamma z^2 = \gamma(1-z)^2\), an ellipse containing \(O = (0,0)\) and \(N = (0,1)\).

This is called the steady-state ellipse [15] since if the control value is fixed to some constant \(u = \bar{u} \in [-1,1]\), the equilibrium of (5) is a point contained on this ellipse. This is illustrated in Fig. 1.

![Fig. 1. The steady state ellipses for oxygenated (blue, solid) and deoxygenated (red, dashed) parameter values. The point marked corresponds to the equilibrium point for the system with \(u = 1/20\).](image)
For a set value $\bar{u} > 0$, the corresponding equilibrium point is
\[
(\bar{y}, \bar{z}) = \left( \frac{-\gamma \bar{u}}{\gamma^2 + \bar{u}^2}, \frac{\gamma^2}{\gamma^2 + \bar{u}^2} \right),
\]
located on the half-hand side of the ellipse. As $\bar{u}$ increases, the corresponding $\bar{q}$ moves from $N$ to $O$ along the ellipse (but not reaching $O$). For $\bar{u} < 0$ the points $\bar{q}$ are symmetric about the $y$-axis.

A second item of interest is the singular manifold, the set of points in which singular arcs are contained. By the discussion of Sec. I-C, the set of order one singular extremals is defined by the relations
\[
\langle p, G \rangle = \langle p, [F, G] \rangle = 0
\]
\[
\langle p, [G, [F, G]] \rangle \neq 0.
\]

By the nontriviality condition of Prop. 1, so $p \neq (0, 0)$, $p \perp G$, and $p \perp [F, G]$ together imply that det$(G, [F, G]) = 0$. By calculation, this is $\gamma(\gamma + 2\delta \bar{z}) = 0$ which is satisfied on the lines $\gamma = 0$ and $\bar{z} = \bar{z}_s := \gamma/(2\delta)$, defining the horizontal and vertical singular lines.

With the physical restriction $2\Gamma \geq \gamma > 0$, we have that $\bar{z}_s = 1$ or $\bar{z}_s < 0$. Thus the horizontal singular line may intersect the Bloch ball nontrivially if $-1 < \bar{z}_s < 0$, or either trivially or not at all otherwise.

To determine the control value, in a manner similar to the elimination of the adjoint $p$ from the determination of the singular manifold, we can reduce $\langle p, [F, [G, G]] \rangle + u(p, [G, [F, G]]) = 0$ to det$(G, [F, [G, G]]) + u_s(G, [F, [G, G]]) = 0$, and this gives the singular control value as a state feedback,
\[
u_s = -\frac{\gamma(2\Gamma - \gamma)y}{2\delta(y^2 - z^2) + \gamma z},
\]
outside the set where det$(G, [F, [G, G]]) = 0$. On the vertical singular line $\gamma = 0$ we have $u_s = 0$, so $q$ approaches $N$ along this line. On the horizontal singular line $z = \bar{z}_s$, the control is $u_s = \gamma(2\Gamma - \gamma)/(2\delta y)$ which yields
\[
y' = -\Gamma y - \frac{\gamma^2(2\Gamma - \gamma)}{4\delta^2 y^2}, \quad \bar{z} = 0,
\]
and we have sgn($\gamma y$) = sgn($\gamma$) so the direction is toward $y = 0$ along this line, and $|u_s| \to \infty$ as $y \to 0$.

Finally, the Legendre–Clebsch condition is checked on these singular lines to determine optimality of singular extremals. Again the adjoint $p$ can be eliminated from (4) since it can be shown that
\[
\text{sgn} \langle p, [G, [F, G]] \rangle = \text{sgn} \det(G, F) \det(G, [F, [G, G]]),
\]
and this sign is positive on the horizontal singular, and on the vertical singular line it is sgn($\gamma y^2(1 - z)(\gamma - 2\delta z)$) which is positive for $z > \bar{z}_s$. Therefore singular extremals are small-time minimizing on the horizontal singular line and on the vertical singular line for $z > \bar{z}_s$.

The singular lines, direction of flow, and optimality are qualitatively shown in Fig. 2.

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**B. The saturation problem: extremal trajectories**

Further analysis [13], [14] is required to give the full optimal control synthesis, but the main components as given above. The locally-optimal trajectory is a concatenation of arcs $\sigma_s, \sigma_b, \sigma_r, \sigma_l$ (symmetrically, $\sigma, \sigma_b, \sigma_r$) where $\sigma_s$ denotes a bang arc with $u = +1$, $\sigma_b$ denotes a singular arc along the horizontal singular line, and $\sigma_r$ denotes a singular arc along the vertical singular line. Some of these arcs may be empty, in particular when the horizontal singular line does not intersect the state space.

The horizontal singular line cannot be followed to its intersection with the vertical singular line due to the control bound, necessitating the bang arc joining $\sigma_b$ and $\sigma_r$. This must be chosen appropriately to ensure this connection is a *bridge*: a bang arc such that the singular-bang-singular concatenation is an optimal extremal. This sequence is qualitatively shown in Fig. 3.

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**C. Results on the control problem**

Several works have treated the control problem by saturation from analytical [6], [8], [7], numerical [1], [2], and experimental [5] points of view. As stated earlier, the saturation problem is of interest in both the analytical and numerical approaches. This motivates studying the single-spin return trip problem and the role it plays in the two-spin return trip problem.

The analytic approaches show that the problem has considerable depth, and closed-form solutions are not reached.

Recent work [16] addresses the single-input contrast problem by saturation under three separate numerical approaches: an indirect shooting and homotopy method, a direct method on a finite-dimensional approximation of the problem, and a method which poses the problem in the form of linear matrix inequalities. The first two methods are used to construct controls achieving locally-minimal cost, and the third method is used to determine a global value for the optimal cost.
Agreement is seen between the first two methods, and the estimate of the globally optimal cost obtained by the third method demonstrates that the controls produced by the first methods are strong candidates for global optima.

III. THE ROUND-TRIP CONTRAST PROBLEM

As in the contrast problem, it is useful to first study the problem in the single-spin case to gain insight into the full problem.

The goal of the return trip of a round trip is to reset the problem to its initial point in minimum time. The natural choice for this point is \( N \), but this point cannot be reached in finite time by extremal trajectories: singular extremals approach it asymptotically, and bang extremals intersecting \( N \) originate outside the state space. We therefore need to consider a point which can be reached in finite time. For this we take a point \( N' \). In the single-spin case, \( N' \) is taken on the steady-state ellipse. This choice allows \( N' \) to maintain its role as the globally attractive equilibrium point of the system under some constant \( v = \dot{u} \), given in (6), and is a point reachable by an extremal trajectory since the vertical singular line can be followed to a point where a bang arc will intersect \( N' \). This choice also has no qualitative impact on the “forward” problem since the control strategy (bang-singular-bang-singular) is identical. In the two-spin case, \( N' \) is similarly taken as a point on the pair of steady-state ellipses.

A. Single-spin return trip

Here we consider the time-minimal transfer from a point \( q^0 \in B \) to \( N' \). We assume only that \( ||q^0|| < ||N'|| \) in order to simplify the discussion and to match the conditions of the problem.

The steady-state ellipse and singular lines introduced in Sec. II-A apply here as well. With these in mind, the possibilities for extremal trajectories (concatenations of bang and singular arcs) are greatly narrowed. The terminal point \( N' \) must be reached by a bang arc from the vertical singular line, so the final two arcs of the trajectory are singular-bang. This singular line can be reached in two ways: either by a bang arc, or by a bang-singular-bang sequence used in the saturation problem (shown in Fig. 3). As shown in the following proposition, the choice is a bang arc to reach the line \( y = 0 \) with \( z \geq 0 \).

**Proposition 2:** The time-minimal transfer from a point \( q^0 = (y^0, z^0) \) to \( N' \) on the steady-state ellipse is of the form bang-singular-bang (where some arcs may be empty). The initial bang, \( u = +1 \) if \( y^0 \geq 0 \) and \( u = -1 \) if \( y^0 < 0 \), is used to reach the line \( y = 0 \) at the point \( q^1 = (0, z^1) \) such that \( z^1 \geq 0 \). The singular control \( u = 0 \) is used to follow the vertical singular line to the point \( q^2 = (0, z^2) \) such that a bang arc can be used to transfer from \( q^2 \) to \( N' \).

**Proof:** It remains to justify that the first step is a single bang to reach \( q^1 \).

If the parameters are such that the horizontal singular line \( z = z_s \) does not intersect \( B \) nontrivially, then initial bang arc is the only extremal arc which reaches the line \( y = 0 \). The choice of the sign of \( u \) as described is clear since it is time-minimal to reach \( y = 0 \) in the upper half-plane as opposed to the lower, and we are done.

Otherwise the parameters are such that \(-1 < z_s < 0\), and in this case the alternate candidate to the proposed single bang is a bang-singular-bang-singular (BSBS) sequence to reach \( q^1 \). Call the arrival of this trajectory to the line \( y = 0 \) \((0, z')\), and we have that \( z' < z^1 \). From here, \( y = 0 \) is followed upward to \((0, z^2)\), passing through \((0, z^1)\). After reaching \((0, z^2)\) the trajectories are identical, so let us show that the time to reach \((0, z^1)\) by a single bang arc is shorter in time duration than this BSBS sequence.

We will show this by comparing the transfer time in the vertical direction. Let us make this comparison along each arc of the BSBS sequence. First consider the case where \( z^0 < z_s \). In the BSBS strategy, the initial bang arc is used to reach \( z = z_s \), identical to the single bang strategy. The BSBS strategy then follows the line \( z = z_s \), along which \( \dot{z} = 0 \). By comparison, the single bang strategy maintains the bang arc with \( \dot{z} > 0 \), in other words not vertically pausing at \( z = z_s \).

Next the BSBS sequence uses a bang arc to reach \( y = 0 \), and here \( \dot{z} > 0 \). In terms of the \( z \)-coordinate, it goes from \( z_s \) to some \( \hat{z} \) along this arc under \( \dot{z} = \gamma(1 - z) + |y| \). The \( z \)-coordinate of the single-bang trajectory moves from \( z_s \) to \( \hat{z} \) with the same \( \dot{z} = \gamma(1 - z) + |y| \), but at each point along its trajectory \( y \)-coordinate is greater in magnitude than the corresponding \( y \)-coordinate of the BSBS strategy (it has followed the horizontal line inward, and the trajectories cannot cross), so in the single-bang case, \( \dot{z} \) is greater than in the BSBS case.

Finally, the BSBS sequence then follows \( y = 0 \) upward with \( u = 0 \) to reach \((0, z^1)\), with \( \dot{z} = \gamma(1 - z) \). As the single-bang \( z \)-coordinate moves through these same \( z \)-values, it has
TABLE I
TIME-OPTIMAL TRANSFER TIME $T^*$ COMPARED WITH THE HEURISTIC TRANSFER TIME $\bar{T}$ TO $N'$ WITH $\bar{u} = 0.1$ FOR PARAMETERS OF DEOXYGENATED BLOOD, WITH THE GIVEN INITIAL POINT $q^0$.

<table>
<thead>
<tr>
<th>$q^0$</th>
<th>$T^*$</th>
<th>$\bar{T}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(0.4)$</td>
<td>6.69</td>
<td>58.73</td>
</tr>
<tr>
<td>$(0.0)$</td>
<td>26.12</td>
<td>98.44</td>
</tr>
<tr>
<td>$(4, -4)$</td>
<td>29.02</td>
<td>113.08</td>
</tr>
<tr>
<td>$(0.0)$</td>
<td>35.04</td>
<td>114.93</td>
</tr>
<tr>
<td>$(0, 0)$</td>
<td>39.40</td>
<td>101.68</td>
</tr>
</tbody>
</table>

Although work on the general return-trip problem has not been pursued in depth, we again have that a great deal of its analysis is shared with the contrast problem.

In the bang-singular-bang sequence, the duration of the singular arc is typically far longer than the two bang arcs. This presents a tradeoff: a choice of $\bar{u}$ with larger magnitude will move $N'$ further down and require less time along $y = 0$, decreasing the return trip time. However, moving $N'$ further away from $N$ is restrictive in the sense that the state space is somewhat shrunk. For example, in the contrast problem one spin is sent to the origin and magnitude of the second is maximized, but this magnitude is generically no greater than the magnitude of the initial point, i.e., $\|N\|$ is an upper bound on achievable contrast. Numerical values for this tradeoff are given in Table II.

B. Two-spin return trip

As the return trip following a sequence to produce contrast by saturation, we assume that the initial point of the return trip is of the form $q^0 = (0, 0, y_2^0, z_2^0)$, and the goal is to reach $N'$ corresponding to a given $\bar{u}$ in minimum time. In this problem, it is difficult even to construct extremal trajectories from a given $q^0$ to some $N'$. To enable preliminary work on the problem, we consider a relaxed terminal condition. This terminal condition is that at the final time $T$, writing $N' = (q_1, q_2)$, we have $q_1(T) = \bar{q}_1$ and $\|q_2(T) - \bar{q}_2\| < \epsilon$, where $\epsilon$ is a prescribed tolerance. This choice maintains $q_1$ as the spin that is carefully controlled—in the contrast by saturation it is controlled exactly to the origin, and so we maintain it as the spin controlled exactly back to $\bar{q}_1$. Similarly, $q_2$ is somewhat less tightly controlled in the contrast problem and in this relaxed terminal condition.

A first heuristic is to use the time-optimal control in the single-spin case for $q_1$, then apply $u = \bar{u}$ (maintaining $q_1$ at $\bar{q}_1$) until the terminal condition is satisfied for $q_2$. This fills a role dual to the saturation problem: in the contrast problem by saturation, a limit case is when the final time is fixed to the minimum time required for the first spin to reach the origin; in the return-trip problem a limit case is that $T \to T_{\text{min}}$ as $\epsilon$ increases, where $T_{\text{min}}$ is the duration of the time-minimum transfer from $O$ to $N'$ in the single-spin case for $q_1$.

Numerical results of this heuristic are illustrated in Fig. 5. Values for a variety of initial points are given in Table III.

IV. CONCLUSION

We see parallels between the contrast problem and the return-trip problem. In both cases, the respective single-spin sub-problem has a relatively straightforward solution, and in particular these problems share geometric characteristics. Although work on the general return-trip problem has not been pursued in depth, we again have that a great deal of its analysis is shared with the contrast problem.

Fig. 4. Time-optimal (blue) versus the heuristic (red, dashed) transfer to $N'$. The physical parameters are of deoxygenated blood, and the values are $\bar{u} = 1$ and $q^0 = (4, -4)$. The duration of the time-optimal transfer is $T = 29.02$ compared to the heuristic transfer duration of 113.08. The steady-state ellipse is shown for reference.
requires refinement of the existing numerical techniques.

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REFERENCES


TABLE III

<table>
<thead>
<tr>
<th>$q_2$</th>
<th>$T_{\text{min}}$</th>
<th>$T$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0.25, 0.25)</td>
<td>83.65</td>
<td>168.59</td>
</tr>
<tr>
<td>(0.25, –0.25)</td>
<td>83.65</td>
<td>163.74</td>
</tr>
<tr>
<td>(0.5, 0)</td>
<td>83.65</td>
<td>166.30</td>
</tr>
<tr>
<td>(0, –0.5)</td>
<td>83.65</td>
<td>160.84</td>
</tr>
</tbody>
</table>

Fig. 5. The heuristic return trip from initial point $q^0 = (0, 0.25, 0.25)$ for deoxygenated and oxygenated blood parameters and $\tilde{u} = 0.05$. The time-minimal control is applied to the first spin, reaching $q^1_2$, followed by $u \equiv \tilde{u}$ until $\|q_2(T) - q_2\| < \varepsilon$ with $\varepsilon = 10^{-2}$. The time to reach $q_2^1$ is 83.65 and the final time is 168.59.

Future work will extend the preliminary results of the two-spin return trip. Numerical methods similar to those applied to the contrast problem are an obvious and promising first step. In particular, a crucial point is to construct an extremal trajectory and verify its optimality status. Another aspect is that if the return-trip terminal condition is relaxed so that the second spin is only within a neighborhood of its previous initial point, the next iteration will produce a slightly different trajectory. This leads to considerations such as either preparing a sequence of contrast-producing controls, which may be computationally prohibitive, or designing one that is robust with respect to such perturbations which