A Fault Prediction Scheme for Takagi-Sugeno Fuzzy Systems with Immeasurable Premise Variables and Disturbance

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Abstract—As explained in the literature, it is very hard to measure premise variables of a Takagi-Sugeno (TS) fuzzy system. Therefore, in this paper, a fault detection and prediction (FDP) scheme is designed for a class of TS fuzzy systems with immeasurable (unknown) premise variables and external disturbances. A fault detection (FD) observer is designed to approximate the system output and the premise variables. Subsequently, a FD residual is generated by comparing the observer output with respect to the system output. The FD residual is evaluated to detect any faults in the system. Further, time-to-failure (TTF) of the TS fuzzy system is obtained by using a mathematical equation. Note the parameter update law and TTF scheme utilize the approximated premise variables since they are not measurable. Stability of the fault detection and TTF prediction results are verified using Lyapunov theory. Finally, a simulation study using a truck-trailer system is presented to verify the theoretical claims.

Keywords: Fuzzy systems, immeasurable premise variables, fault detection, prognostics.

I. INTRODUCTION

Takagi-Sugeno (TS) fuzzy models are extensively used for fault detection and diagnosis due to their superior capability in approximating nonlinear dynamics [1-11]. Some of the previously reported model-based fault detection (FD) schemes for TS fuzzy systems with measurable premise variables are given in [1-7]. In [1-2] observer-based techniques are proposed, H_\inf-based design in [3], and sliding-mode observer in [4]. In practice, it is very hard to measure premise variables [8-11].

Also, fault detection schemes developed for TS fuzzy systems with measurable premise variables cannot be directly applied to systems with immeasurable premise variables [8-11]. Since the observer design and their associated analytical results have to be modified. Therefore, in [8], a robust observer based scheme is proposed. In [9, 10], an observer based fault detection is developed with Linear Matrix Inequality (LMI) conditions. In all of the above methods, only fault detection is addressed.

On the other hand, prognostics (remaining useful life) schemes have been developed for non-fuzzy systems [12-16]. In our previous work [18], model-based fault detection and prognostics scheme was developed for TS fuzzy systems with measured premise variables. As explained earlier, it is very hard to measure premise variables of fuzzy systems [8-11]. Therefore, a dedicated model-based FDP scheme is developed for TS fuzzy systems with immeasurable premise variables.

In this paper, the fault detection observer design consists of an OLAD (online approximator in discrete time), which is used to learn the fault occurring in the TS fuzzy system. A unique parameter update law using the estimated premise variables is developed to tune the OLAD in real-time. The estimated fault magnitude is used for calculating time to failure (TTF).

In order to calculate TTF, a mathematical equation is developed using the approximated premise variables. As explained in [18], a fault typically affects the system parameter. Evolution of the faulty system parameter is calculated using OLAD of the FD observer. Consequently, the estimated fault magnitude is projected against its failure limit to determine remaining useful life (or TTF). Unlike [13-16], the proposed TTF scheme does not require failure data for training.

Important differences between the FDP scheme proposed in this paper and previous publications [1-11, 18] are explained as follows: 1) The proposed FDP scheme can do both fault detection and prediction for TS fuzzy systems with immeasurable premise variables, 2) FD observer (with approximated premise variables) has an Online Approximator in Discrete-time (OLAD) to estimate the fault magnitude in real-time, 3) An analytical method is presented for choosing FD observer gain which separates disturbances from FD residual, 4) A simple method is devised to determine time-to-failure (remaining useful life) in real time (i.e., without using any offline training), and 5) Analytical results using approximated premise variables are presented to guarantee the stability of the proposed fault detection and prediction scheme.
In the next section, the system under investigation is explained in detail.

II. SYSTEM DESCRIPTION

As explained in [5], Takagi-Sugeno (TS) fuzzy models are widely used for approximating nonlinear systems. In the TS approach, IF-THEN conditions are used to capture the local linear behavior of a system. Consequently, all the local linear fuzzy models are aggregated to approximate the overall nonlinear system dynamics. Consider the following \( l \)th rule of the TS fuzzy model

\[
\text{Rule } i: \text{ If } \xi(k) \text{ is } \Delta_i \text{ THEN }
\]

\[
z_i(k+1) = S_i z_i(k) + H_i u(k) + D_i \omega(k) + g_i(y, \theta)
\]

\[
y(k) = C_i z_i(k)
\]

where \( i = 1, 2, \ldots, M \). \( M \) is the number of IF-THEN rules, \( \xi(k) \) is the premise variable, \( \Delta_i \) is the fuzzy set, \( z_i(k) \in \mathbb{R}^s \) represents the system state, \( y(k) \in \mathbb{R}^r \) is the measured output, \( u(k) \in \mathbb{R}^n \) is the system input, \( \omega(k) \) represents the unknown bounded disturbances. Further, \( S_i, H_i, C_i \), and \( D_i \) are known system matrices of appropriate dimensions, \( g_i(y, \theta) \) represents the fault dynamics. In addition, \( g_i(y, \theta) = \Omega_i (k-k_0) \theta' f_i(y(k)) \) and \( \theta \in \mathbb{R}^{m \theta} \) is the unknown fault magnitude, \( \Omega_i (k-k_0) \) is the time profile of the fault dynamics, which could be incipient (slowly growing) or abrupt (sudden) in nature. The fault basis function \( f_i \in \mathbb{R}^n \) could be any continuous mathematical function.

Considering \( M \) fuzzy rules, the nonlinear system could mathematically be represented as shown below

\[
z_i(k+1) = \sum_{i=1}^{M} \mu_i(\xi(k)) (S_i z_i(k) + H_i u(k) + D_i \omega(k) + g_i(y, \theta))
\]

\[
y(k) = \sum_{i=1}^{M} \mu_i(\xi(k)) C_i z_i(k) = C z(k)
\]

where \( \mu_i(\xi(k)) \) is the fuzzy weighting function. The time profile of the fault dynamics in (2) is given by

For incipient faults:

\[
\Omega_i(\tau) = \begin{cases} 
0, & \text{if } \tau < 0 \\
(1-e^{-\tau/\tau_i}), & \text{if } \tau \geq 0
\end{cases}, \quad i = 1, \ldots, M
\]

where \( \tau_i > 0 \) is an unknown constant that represents the rate at which the fault grows.

For abrupt faults:

\[
\Omega_i(\tau) = \begin{cases} 
0, & \text{if } \tau < 0 \\
1, & \text{if } \tau \geq 0
\end{cases}, \quad i = 1, \ldots, M
\]

where \( \tau \) is the time of fault occurrence. The weighting function \( \mu_i(\xi(k)) \) in (2) is a function of the premise variable \( \xi(k) \), which is the state variable, i.e. \( \xi(k) = z_i(k) \). Also, the weighting function satisfies

\[
0 \leq \mu_i(\xi(k)) \leq 1, \quad \sum_{i=1}^{M} \mu_i(\xi(k)) = 1 \quad \forall k \in \mathbb{R}^n
\]

The TS fuzzy system in (2) could be subjected to one or more state faults (also known as component faults). As shown in (2), the case of multiple faults is very challenging, but it is still addressed in this paper. In the next section, the fault detection scheme is introduced.

III. FAULT DETECTION SCHEME

Mathematical representation of the fault detection observer used for monitoring and detecting faults in the TS fuzzy system with immeasurable premise variables in (2) is given below

\[
\dot{z}_i(k+1) = \sum_{i=1}^{M} \mu_i(\hat{\xi}(k)) (S_i \hat{z}_i(k) + H_i u(k) + \hat{g}_i(y, \hat{\theta}(k)) + K_i (y(k) - \hat{y}(k)))
\]

\[
\hat{y}(k) = C z(k)
\]

where \( \hat{z}_i(k) \in \mathbb{R}^s \) represents the estimated system state, \( \hat{g}_i(y, \hat{\theta}(k)) \) is the weighting function which satisfies the convex property given in equation (3), \( \hat{y}(k) \in \mathbb{R}^r \) is the measured output, \( K_i \in \mathbb{R}^{m \theta} \) is the user-defined observer gain, \( \hat{g}_i(y, \hat{\theta}(k)) = \hat{\theta}' f_i(y(k)) \) is the OLAD and \( \hat{\theta}(k) \) is the estimated fault magnitude. Unlike [18], later in this paper, a suitable parameter update law is developed using the estimated premise variables to tune \( \theta \in \mathbb{R}^{m \theta} \) in real-time. Thus the fault evolution could be estimated online without any apriori offline training. Remember, the premise variables are not measurable in eq. (2).

Next, define the state residual as \( e_i(k) = y_i(k) - \hat{y}_i(k) \) and output residual as \( e_q(k) = y(k) - \hat{y}(k) \). Due to the presence of disturbances in (2), a suitable fault detection residual is introduced, which would be very sensitive towards the fault, but less sensitive towards disturbances.

The state and output residual dynamics from (2) and (4) prior to the occurrence of a fault is given by

\[
e_i(k+1) = \sum_{i=1}^{M} \mu_i(\hat{\xi}(k)) (S_i z_i(k) + H_i u(k) + D_i \omega(k))
\]

\[
- \sum_{i=1}^{M} \mu_i(\hat{\xi}(k)) (S_i \hat{z}_i(k) + H_i u(k) + K_i (y(k) - \hat{y}(k)))
\]

\[
e_q(k) = Ce_q(k)
\]

Add and subtract

\[
\sum_{i=1}^{M} \mu_i(\hat{\xi}(k)) (S_i z_i(k) + H_i u(k) + D_i \omega(k))
\]

to the above equation, therefore, we have

\[
e_i(k+1) = \sum_{i=1}^{M} \mu_i(\hat{\xi}(k)) - \mu_i(\hat{\xi}(k)) (S_i z_i(k) + H_i u(k) + D_i \omega(k))
\]

\[
+ \sum_{i=1}^{M} \mu_i(\hat{\xi}(k)) (S_i \hat{z}_i(k) + H_i u(k) + K_i (y(k) - \hat{y}(k)))
\]

\[
e_q(k) = Ce_q(k)
\]

The following lemma is proposed for choosing the fault detection observer gain \( K_i \).

**Lemma 1** [17]: Let \( Q \) be a user-defined constant matrix. Then the fault detection observer gain \( K_i \) could be calculated if either of the following conditions are satisfied
1) \( \sum_{i=1}^{M} \mu_i(\tilde{z}(k)) C \left( \sum_{i=1}^{M} \mu_i(\tilde{z}(k)) D_i \right) = 0 \)
2) \( \sum_{i=1}^{M} \mu_i(\tilde{z}(k)) C \left( \sum_{i=1}^{M} \mu_i(\tilde{z}(k))[S_i - K_i C] \right) = 0 \)

The following criteria is introduced for detecting faults in the given TS fuzzy system

\[
\frac{e_f(k)}{e_{th}} < \text{no faults} \quad \frac{e_f(k)}{e_{th}} \geq \text{faults}
\]

The FD threshold \( e_{th} \) is calculated by using the following equations

\[
e_\theta(k) = \left( \sum_{i \in e} e_i^2(k) \right)^{1/2}
\]

where \( k_0 \) denotes initial evaluation time instant and \( k^* \) is final evaluation time. The above equations are commonly used in the fault detection literature [12]. The fault detection residual will be less than the threshold \( e_{th} \) prior to the occurrence of a fault. When the fault occurs, the fault detection residual would exceed the threshold and the fault would be detected. Note if the fault detection observer gain \( K_i \) is properly chosen, the disturbance could be decoupled from the fault detection residual. Consequently, the fault detection residual will be zero prior to the occurrence of a fault, and in that case, there is no need to apply a fault detection threshold. This would be illustrated in the simulation results presented later in this paper.

After a fault occurs, the OLAD (\( g_i(y, \hat{\theta}(k)) = \hat{\theta}^f(k) f_i(k) \)) in the fault detection observer in (4) is initiated to learn the unknown fault magnitude in real-time. Thus the fault evolution is continuously estimated.

The state and output residual after the detection of a fault would be given by the following equation

\[
e_f(k+1) = \sum_{i=1}^{M} \left( \mu_i(\tilde{z}(k)) - \mu_i(\tilde{z}(k)) \right) [S_i z_i(k) + H_i \mu_i(k) + D_i \alpha_i(k) + y_i(k, \theta) - y_i(k, \hat{\theta})] = \sum_{i=1}^{M} \mu_i(\tilde{z}(k)) [S_i e_i(k) + D_i \alpha_i(k) + \psi_i(k)]
\]

where \( S_i = S_i - K_i C_i \), \( \psi_i = \hat{\theta}^f(k) f_i(k) \), \( \hat{\theta} = \theta(\hat{\theta}) \) is the parameter estimation error. Note the OLAD in the fault detection observer given in (4) is activated only after the detection of a fault. The parameter estimation error term in (9) measures the quality of the OLAD in learning the actual fault dynamics. In other words, if the estimation error \( \| \hat{\theta}(k) \| \) remains bounded, the fault dynamics will be approximated satisfactorily by the OLAD. But to tune the OLAD parameter \( \hat{\theta}(k) \), the following update law is introduced using the approximated premise variables

\[
\hat{\theta}(k+1) = \hat{\theta}(k) + \alpha \left( \sum_{i=1}^{M} \mu_i(\tilde{z}(k)) f_i(k) \right) e_i^f(k) + \frac{\alpha}{\gamma} \left( \sum_{i=1}^{M} \mu_i(\tilde{z}(k)) f_i(k) \right) e_i^f(k)
\]

where \( \alpha > 0 \) is the learning rate and \( \gamma > 0 \) is a design constant. Additionally, \( \gamma B_k \in \mathbb{R}^{n \times \alpha} \) is a constant matrix chosen by the user such that \( \| \gamma B_k \| \leq \rho \), \( \rho > 0 \), and \( I \) is an identity matrix of appropriate dimension. Therefore, for the \( M \) fuzzy models, the fault magnitude is estimated by using the update law given in (10).

In order to guarantee that the proposed fault detection observer in (4) could learn the fault dynamics online, the following theorem is introduced. Note such detailed analytical results were not presented in the literature [1-11, 18] for TS fuzzy systems with immeasurable premise variables.

Theorem 1 (FD Observer Stability Analysis): Consider the FD observer given in (4) for monitoring the TS fuzzy system given in (2). Further, the update law given in (10) is used for tuning the parameters of the OLAD. Then the output residual \( e_f(k) \) and the parameter estimation error \( \hat{\theta}(k) \) are uniformly ultimately bounded (UUB).

Proof: Refer to Appendix.

Unlike [18], in the above theorem, only the approximated premise variables are used, therefore, the analytical results are complicated, but is still addressed in this paper. The above theorem analytically guarantees that the fault magnitude \( \hat{\theta} \) will be estimated satisfactorily by the OLAD (\( \theta \)). Therefore, a method is proposed to continuously monitor fault growth, which is required to estimate time-to-failure or remaining useful life. In the next section, the TTF scheme is introduced.

IV. PREDICTION SCHEME

In this paper, a method is proposed where the online estimation of the fault magnitude is used to determine TTF. The estimated fault magnitude is projected against their corresponding failure limit to determine time-to-failure. In the following theorem, a mathematical equation is developed to determine TTF by using the estimated fault magnitude and failure limit.

Theorem 2 (Time to Failure): Consider the system described in (2) with a fault. The TTF for the \( j \)th system parameter at the \( k \)th time instant using the approximated premise variables is given by the following equation

\[
\log \left[ \begin{array}{c}
\theta_{\bar{\alpha}(\hat{\theta}(k))} \\
\theta_{\bar{\alpha}(\hat{\theta}(k))}
\end{array} \right] = \alpha \left( \sum_{i=1}^{M} \mu_i(\tilde{z}(k)) f_i(k) \right) e_i^f(k) + k_n
\]

where

\[
k_n = \log \left[ \begin{array}{c}
\sum_{i=1}^{M} \mu_i(\tilde{z}(k)) f_i(k) e_i^f(k) B_k
\end{array} \right] + k_n
\]

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where \( k_{fij} \) and \( k_{0ij} \) are respectively the estimated TTF and the time instant when the prediction is estimated. \( \theta_{ij}^{\text{max}} \) is the maximum value (failure threshold) of the system parameter, and \( \theta_{ij}^{0} \) is the value of the system parameter at the \( k_{ij}^{0} \) time instant.

**Proof:** Using (10) and following steps given in [18], one can complete the mathematical proof.

Unlike [18], equation (11) utilizes approximated premise variables. At each time instant after the detection of a fault, TTF could be determined using equation (11). The following algorithm is proposed to recursively estimate TTF after the detection of a fault.

**Algorithm:** The following steps outline the use of equation (11) in determining TTF

**Step 1:** Fault detected, \( k_{0ij} = k_{dt} \) (time of fault detection).

**Step 2:** Activate OLAD.

**Step 3:** Calculate \( \left( \sum_{i=1}^{N} \mu_i(\hat{z}_i(k_i^n)) \right) \text{f}_j \) and \( \hat{\theta}_j(k_{0ij}^{th}) \) at the \( k_{0ij}^{th} \) instant. In particular, \( e_j(k_{0ij}^{th}) \) is obtained from eq. (9), whereas \( \hat{\theta}_j(k_{0ij}^{th}) \) and \( \left( \sum_{i=1}^{N} \mu_i(\hat{z}_i(k_i^n)) \right) \) are available.

**Step 4:** Use values derived from Step 3 in equation (11) to calculate TTF for the \( j^{th} \) fault.

**Step 5:** \( k_{TTF}^{th} = \min(k_{fij}^{th}) \), minimum of all the estimated TTF's when the system in (2) is subjected to multiple faults.

**Step 6:** If \( k_{TTF}^{th} = k_{jit} \) (time of failure)

- System unsafe to operate, stop.
- Else

\[ k_{ij}^{0} = k_{0ij}^{th} + 1 \text{, and Go to Step 3} \]

The proposed prediction scheme can handle multiple (simultaneous) faults. Therefore, TTF is determined for each fault and by taking a minimum of all the estimated TTFs, the overall TTF for the system could be determined. In summary, the following equation is used to determine TTF for multiple faults

\[ k_{TTF} = \min(k_{fij}^{th}) \quad l = 1, \ldots, n, j = 1, \ldots, p \] (12)

In the next section, a simulation case study is presented to illustrate the proposed fault-detection and prediction scheme.

V. SIMULATION RESULTS

In this simulation study, a truck-trailer system is considered for demonstrating the proposed fault-detection and prediction scheme. From [7], the following Fuzzy model of the truck-trailer system is obtained

\[ z_s(k+1) = \sum_{i=1}^{N} \mu_i(\hat{z}_i(k)) (S_i z_i(k) + H_i u(k) + D_i \alpha(k) + g_i(k, \theta)) \]

\[ y(k) = C z_s(k) \] (13)

where \( z_s(k) = [z_1(k), z_2(k), z_3(k)]^T \), \( z_i(k) \) is the angle difference between truck and trailer (in radians), \( z_i(k) \) is the angle of trailer (in radians), \( z_i(k) \) is the vertical position of rear of trailer (in m), \( u(k) \) is the steering angle (in radians), and \( y(k) \) is the output. We consider a two-rule fuzzy model, therefore

\[
\begin{bmatrix}
1 - \frac{VT}{l_{\text{trailer}}} & 0 & 0 \\
\frac{l_{\text{trailer}}}{VT} & 1 & 0 \\
\frac{2l_{\text{trailer}}}{VT} & 1 & 0.01(VT)^2 & 0.01VT & \pi & 1
\end{bmatrix}
\]

\[
H_1 = H_2 = \begin{bmatrix}
VT \\
0 \\
0
\end{bmatrix}
\]

\[
C_1 = C_2 = \begin{bmatrix}
1 & 0 \\
0 & 0 \\
1
\end{bmatrix}
\]

\[
D_1 = D_2 = \begin{bmatrix}
0 & 0.1 & 0.1
\end{bmatrix}
\]

where \( l_{\text{trailer}} \) is the length of trailer (in m), \( l_{\text{truck}} \) is the length of truck (in m), \( \nu \) is the constant speed of backing up, and \( \tau \) is the sampling time. Here we define \( \theta(k) = (VT / 2l_{\text{trailer}})z_i(k) + z_i(k) \cdot z_i(k) \) is observed variable, but \( z_i(k) \) is not measurable; therefore, we use \( z_i(k) \) to calculate \( \theta(k) \). In this study, we consider a trapezoidal membership function. Additionally, \( \sigma(k) \) is considered to be a random white noise.

We assume a time-varying actuator fault that is mathematically described by

\[ g_i(k, \theta) = g_i(k, \theta) = 1.5(1 - e^{-0.0004-k_i}) \sin(0.0087k) \]

and the incipient fault occurs at \( k_0 = 10 \) sec and the growth rate is \( \mathcal{R}_1 = \mathcal{R}_2 = 0.01 \). In order to detect the fault in the system defined in (13), the following FD observer is used

\[ \dot{\hat{z}}_i(k+1) = \sum_{i=1}^{N} \mu_i(\hat{z}_i(k)) (S_i \hat{z}_i(k) + H_i u(k) + \dot{g}_i(k, \hat{\theta}(k)) + K_i (y(k) - \hat{y}(k))) \]

\[ \hat{y}(k) = C z_s(k) \] (14)

where \( \hat{z}_i(k) = [\hat{z}_1(k), \hat{z}_2(k), \hat{z}_3(k)]^T \) is the estimated system states,

\[ \dot{g}_i(k, \hat{\theta}(k)) \]

is the OLAD defined by

\[ [\hat{\theta}_1(k) \quad \hat{\theta}_2(k) \quad \hat{\theta}_3(k)]^T \sin(0.0087k) \]

and the OLAD parameters are tuned using the update law given in (10) with \( \alpha = 0.28 \) and \( \gamma = 0.00002 \). Using results of Lemma 1, gains of the FD observer are calculated as given below

\[ K_1 = [-4.33 \quad 4.05 \quad -0.44]^T \]

and

\[ K_2 = [-2.55 \quad 0.65 \quad -0.62]^T \]

In order to detect fault in (13), a fault detection residual is generated as shown in Fig. 1. As shown in the figure, the FD residual stays at zero prior to the fault though there is disturbance in the system. The reason is due to the results of Lemma 1, which ensures that the disturbances are decoupled.
from the FD residual. Since the FD residual is decoupled from the disturbance, there is no need to use a FD threshold. Therefore, as soon as the fault occurs, the FD residual becomes non-zero; thereby the fault is detected instantaneously.

**Fig. 1:** Fault detection residual.

The fault magnitude is estimated online as shown in Fig. 2, and the fault evolution is successfully captured by the OLAD in (14). Using the estimated fault magnitude in (11) and a failure threshold of 0.2 units, TTF is estimated as shown in Fig. 3. Overall, the TTF result is found to be satisfactory. Note no failure data is used for determining TTF and only equation (11) is needed, hence not computationally intensive. Also, to best of author’s knowledge there is no previously reported FDP scheme for TS fuzzy systems with immeasurable premise variables. Therefore, the above simulation results are unique.

**Fig. 2:** Actual and estimated fault magnitudes, and failure threshold.

**Fig. 3:** Estimated time-to-failure.

VI. CONCLUSIONS

A model-based fault detection and prediction scheme was developed for a class of TS fuzzy systems with immeasurable premise variables. The analytical and simulation studies show that the proposed FD observer successfully detects and learns online (using the OLAD) the unknown fault. Fault estimation results were used in the time-to-failure prediction, and were found to be very satisfactory. Possible future research is to design a FDP scheme for networked control systems.

APPENDIX

Proof of Theorem 1: Unlike [18], the analytical results presented below are complex due to the use of approximated premise variables. Consider the following Lyapunov function candidate

\[ V = e^T(k)e(k) + \frac{1}{\alpha} tr(\hat{\theta}^T(k)\hat{\theta}(k)) \]

First difference of the above equation is given by

\[ \Delta V = \frac{1}{\Delta T} tr(\hat{\theta}^T(k + 1)\hat{\theta}(k + 1) - \hat{\theta}^T(k)\hat{\theta}(k)) \]

(A.1) Consider \( \Delta V_1 \) from (A.1), substitute the state residual dynamics from (9) to obtain the following equation

\[ \Delta V_1 = \sum_{i,j=1}^{M} (\mu_i(\tilde{\xi}(k)) - \mu_j(\tilde{\xi}(k))) \tilde{S}_i z_i k + H_i \mu(k) + D_0 \alpha(k) + g_i(y_i, \theta)) \]

Next consider \( \Delta V_2 \) of (A.1), substitute (10), after some mathematical manipulation, the following equation is obtained

\[ \Delta V_2 = \frac{1}{\alpha} tr(\hat{\theta}^T(k)\hat{\theta}(k) - 6 \sum_{i} \mu_i(\tilde{\xi}(k)) \| I - \alpha_i f_i(k) \|^2 \hat{\theta}^T(k)\hat{\theta}(k) + 3 \sum_{i} \mu_i(\tilde{\xi}(k)) f_i(k) e_i(k) e^T_i(k) \tilde{B}) \]

(A.3) Combine (A.2) and (A.3), using [19], we take

\[ \sum_{i=1}^{M} \mu_i(\tilde{\xi}(k)) (S_i z_i k + H_i \mu(k) + D_0 \alpha(k)) \leq \eta(k) \]

\( \eta(k) \in \mathfrak{H} \) is a continuous function and apply the following condition

\[ \sum_{i=1}^{M} \mu_i(\tilde{\xi}(k)) (S_i z_i k + H_i \mu(k) + D_0 \alpha(k)) + 3 \sum_{i} \mu_i(\tilde{\xi}(k)) \| I - \alpha_i f_i(k) \|^2 \hat{\theta} \theta + \eta(k) \leq \beta_L \]

\( \beta_L > 0 \), and finally applying Frobenius norm to obtain the following equation

\[ \Delta V \leq -(1 - 3S_{min}^2 - 3C_{max}^2 \rho_{max}^2) \| f_i(k) \| \cdot \beta_L \}

\[ \left( 6 \gamma \sum_{i} \mu_i(\tilde{\xi}(k)) \| I - \alpha_i f_i(k) \|^2 \right) - \frac{3 \gamma f_{max}^2 - 2}{\alpha} \]

(A.4) Therefore, \( \Delta V \leq 0 \) in (A.4) provided the following conditions are satisfied

\[ \| f_i(k) \|_{\alpha} \leq \beta_L \]

or

\[ \| f_i(k) \| \leq \beta_L \]

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\[ \hat{p}(k) = \left( \frac{\hat{\theta}_i}{a} + \sum_{i=1}^{n} \mu_i(\hat{\xi}(k)) \left[ a(\hat{\xi}(k))' f^i(\hat{\xi}(k))' \delta \right] \right) - 3\delta^2 - \frac{3}{a} \sum_{i=1}^{n} \mu_i(\hat{\xi}(k)) \left[ a(\hat{\xi}(k))' f^i(\hat{\xi}(k))' \delta \right] \]

and

\[ S_{\text{max}} \leq \sqrt{\left(1 - 3\varphi_{\text{max}}^2 \sigma_{\text{max}}^2 \rho_{\text{max}}^2 \right) / 3} \times \frac{3 - 9\varphi_{\text{max}}^2}{3 + \sqrt{3 - 9\varphi_{\text{max}}^2}} \times \left( \sum_{i=1}^{n} \mu_i(\hat{\xi}(k)) \left[ a(\hat{\xi}(k))' f^i(\hat{\xi}(k))' \delta \right] \right) \]

\[ \alpha \leq \left\{ \begin{array}{l l} 1/3 \rho_{\text{max}}^2 & \rho_{\text{max}} > 1 \\ 1/3 \rho_{\text{max}}^2 & 0 < \rho_{\text{max}} \leq 1 \end{array} \right. \]

Using results of [chp. 2, 19], the state residual \( \| e_r(k) \| \) and the parameter estimation error \( \| \hat{\theta}(k) \| \) are determined to be uniformly ultimately bounded (UUB). Further, it could be observed that the residual \( \| e_r(k) \| \) is UUB since the state residual \( \| e_r(k) \| \) is bounded. The analytical results guarantee that the estimation of the fault magnitude by the OLAD is stable. As explained in [19], the above analytical conditions could be demonstrated.

REFERENCES


