Control of Hyperbolic PDE Systems with Actuator Dynamics

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Introduction

A simple model of a counter flow heat exchanger is provided by the system of PDEs defined on the interval (0, L) by

\[ \frac{\partial T_1(t,x)}{\partial t} = -v_1 \frac{\partial T_1(t,x)}{\partial x} - \kappa_1 T_1(t,x) + \sqrt{k_1 k_2} T_2(t,x), \]

(1)

Here, \( T_1(t,x) \) represents a (scaled) temperature of a hot fluid flowing from left to right in channel one and \( T_2(t,x) \) represents the (scaled) temperature of a cold fluid flowing from right to left in channel two. At the inflow for channel one, we have a boundary control

\[ T_1(t,0) = u_1(t) \]

(2)

and at the inflow for channel two we have

\[ T_2(t,L) = u_2(t). \]

(3)

The pure boundary control problem for the system (1) - (3) was discussed in [13]. In this paper we consider the case where the control inputs are outputs to a dynamic actuator. This problem is more representative of real systems and although the system is more complex, the resulting control problem is better conditioned.

In order to keep the presentation short we focus on a 1D version of this class of problems, but note that the basic results extend to the general class of hyperbolic systems of the form treated in [5]. Thus, we consider the 1D hyperbolic PDE

\[ \frac{\partial}{\partial t} T(t,x) = -v \frac{\partial}{\partial x} T(t,x) - \kappa T(t,x), \quad 0 < x < L, \quad t > 0, \]

(4)

with boundary condition

\[ T(t,0) = u_0(t). \]

(5)

Here, \( u_0(\cdot) \) is a control input or a design parameter for the system and \( v > 0 \) and \( \kappa \geq 0 \) are constants. The initial data for (4) is given by

\[ T(0,x) = \phi(x) \in L_2(0,L). \]

(6)

The uncontrolled hyperbolic system (4)-(5) defines a dynamical system \( S_0(t) \) on \( X = L^2(0,L) \) generated by the differential operator defined on the domain

\[ \mathcal{D}(S_0) = \{ \phi(\cdot) \in X : \phi(\cdot) \in H^1(0,L), \quad \phi(0) = 0 \}. \]

(7)

by

\[ S_0 \phi(\cdot) = -v \phi'(\cdot) - \kappa \phi(\cdot). \]

(8)

In particular,

\[ \langle S_0 \phi(\cdot), \phi(\cdot) \rangle_{L^2} = \int_0^L -v \phi'(x) \phi(x) \, dx - \kappa \| \phi(\cdot) \|^2 \]

\[ = -\frac{v}{2} \| \phi(\cdot) \|^2 - \kappa \| \phi(\cdot) \|^2 \]

(9)

so that \( S_0 \) generates a dissipative dynamical system and

\[ [S_0(t) \phi(\cdot)](x) = \left\{ \begin{array}{ll} e^{-\kappa t} \phi(x-t), & 0 \leq t \leq L \leq x \leq L \\
0, & x < L < +\infty \end{array} \right. \]

Also note that that this dynamical system is nilpotent (i.e., \( S_0(t) = 0 \) for \( t > L \)).

A straightforward computation show that the adjoint operator \( S_0^* \) is given by

\[ S_0^* \psi(\cdot) = +v \psi'(\cdot) - \kappa \psi(\cdot) \]

(10)

with domain

\[ \mathcal{D}(S_0^*) = \{ \psi(\cdot) \in X : \psi(\cdot) \in H^1(0,L), \quad \psi(L) = 0 \}. \]

The hyperbolic PDE equations used in the above references as models of heat exchangers all have a common structure that is captured by the simple 1D PDE. In particular;

1) The uncontrolled system generates a nilpotent dynamical system.
2) The system operator \( S_0 \phi(\cdot) = -v \frac{\partial}{\partial t} \phi(\cdot) - \kappa \phi(\cdot) \) is non-normal. Also, \( \mathcal{D}(S_0) = \mathcal{D}(S_0^*) \), \( S_0 \neq -S_0^* \) and \( S_0 \neq -S_0^* \).
3) The control problem defined by (4)-(5) does not fall into the standard boundary control theory found in references such as [4], [15], [20] and [21].
4) Although we shall show that the “AVE scheme” (essentially a finite volume method, see [1], [7] and [13]) produces convergent system operators, the convergence of solutions to optimal control problems is not clear cut since this type of system does not fall into the class of systems considered in [20] and [21]. In particular, \( S(t) \) is not a group nor is it an analytic semigroup.
Control problems with non-normal operators are known to cause issues with sensitivity and numerical approximations for optimal control [8]. Moreover, the boundary control problem presents unique theoretical challenges. In [25] this issue is resolved by assuming that the spatial distribution of the actuator spans a small portion of spatial domain. In particular, the control system (4) - (5) is replaced by
\[
\frac{\partial}{\partial t} T(t,x) = -v \frac{\partial}{\partial x} T(t,x) - \kappa T(t,x) + b_\epsilon(x) u_0(t)
\] (11)
with boundary condition
\[
T(t,0) = 0,
\]
where
\[
b_\epsilon(x) = \begin{cases} 
1, & 0 \leq x \leq \epsilon \\
0, & \epsilon < x \leq 1
\end{cases}.
\]
Here, \( \epsilon > 0 \) is “small” and hence \( b_\epsilon(x) \) represents an approximation of the \( \delta \)-function at \( x = 0 \). Thus, the control system (11) becomes a problem with bounded input control operator.

In this paper we take another approach. We start with the system (4)-(5), but also assume that the input \( u_0(t) \) at \( x = 0 \) is the output to a finite dimensional dynamical system (the actuator dynamics) given by
\[
\dot{w}(t) = Aw(t) + Bu_\epsilon(t),
\] (12)
where \( u_\epsilon(t) \) is the actuator control input and
\[
u_0(t) = Cw(t).
\] (13)
This is the approach taken in the papers [10], [11] and [12] for boundary control of parabolic systems with actuator dynamics. Although the combined actuated control system can be more complex, it is often a more realistic model of true systems and, as we see below, the resulting problem no longer has an unbounded input operator.

II. PROBLEM TRANSFORMATION

We start with the basic system
\[
\frac{\partial}{\partial t} T(t,x) = -v \frac{\partial}{\partial x} T(t,x) - \kappa T(t,x),
\] (14)
with Dirichlet boundary inputs
\[
T(t,0) = u_0(t).
\] (15)
Let \( b(\cdot) \) be the solution of the boundary value problem
\[
-v \frac{d}{dx} b(x) - \kappa b(x) = 0, \quad b(0) = 1,
\]
so that
\[
b(x) = \begin{cases} 
1, & \kappa = 0 \\
e^{-(\kappa / 0)x}, & \kappa > 0
\end{cases}.
\]
Make the change of variables
\[
\theta(t,x) = T(t,x) - b(x) u_0(t)
\]
and if one assumes for a moment that \( \dot{u}_0(t) \) exists, then \( \theta(t,x) \) satisfies
\[
\frac{\partial}{\partial t} \theta(t,x) = \frac{\partial}{\partial t} T(t,x) - b(x) \dot{u}_0(t)
\]
\[
= -v \frac{\partial}{\partial x} T(t,x) - \kappa T(t,x) - b(x) \dot{u}_0(t)
\]
\[
= -v \frac{\partial}{\partial x} \theta(t,x) - \kappa \theta(t,x) - b(x) \dot{u}_0(t)
\]
\[
= -v \frac{\partial}{\partial x} \theta(t,x) - \kappa \theta(t,x) - b(x) \dot{u}_0(t)
\]
and since \( v \frac{d}{dx} b(x) + \kappa b(x) = 0 \) it follows that
\[
\frac{\partial}{\partial t} \theta(t,x) = -v \frac{\partial}{\partial x} \theta(t,x) - \kappa \theta(t,x) - b(x) \dot{u}_0(t). \] (16)

Moreover, \( \theta(t,x) \) satisfies the homogenous boundary condition
\[
\theta(t,0) = 0.
\] (17)
We are interested in the case where the input \( u_0(t) \) is given by the output of a finite dimensional differential equation describing the actuator dynamics. We assume that
\[
\dot{w}(t) = Aw(t) + Bu_\epsilon(t),
\] (18)
where \( u_\epsilon(\cdot) : [0, +\infty) \rightarrow \mathbb{R}^m \) is the command to the actuator and
\[
u_0(t) = Cw(t).
\] (19)
Thus, it follows that
\[
\dot{u}_0(t) = C\dot{w}(t) = C[Aw(t) + Bu_\epsilon(t)]
\] (20)
\[
= CAw(t) + CBu_\epsilon(t)
\]
and substituting (20) into (16) yields
\[
\frac{\partial}{\partial t} \theta(t,x) = -v \frac{\partial}{\partial x} \theta(t,x) - \kappa \theta(t,x)
\]
\[
- b(x) [CAw(t) + CBu_\epsilon(t)]
\]
\[
= -v \frac{\partial}{\partial x} \theta(t,x) - \kappa \theta(t,x)
\]
\[
+ [-b(x)CA]w(t) + [-b(x)CB]u_\epsilon(t).
\]
The composite system can now be written as
\[
\frac{\partial}{\partial t} \theta(t,x) = -v \frac{\partial}{\partial x} \theta(t,x) - \kappa \theta(t,x) \] (21)
\[
+ f(x)w(t) + g(x)u_\epsilon(t),
\]
\[
\dot{w}(t) = Aw(t) + Bu_\epsilon(t)
\] (22)
where \( f(x) \triangleq [-b(x)CA] \) and \( g(x) \triangleq [-b(x)CB] \), respectively. Observe that \( f(\cdot) \) and \( g(\cdot) \) belong to \( L^2(0,L) \) and hence the operators \( F : \mathbb{R}^n \rightarrow L^2(0,L) \) and \( G : \mathbb{R}^m \rightarrow L^2(0,L) \) defined by
\[
[Fw](x) \triangleq f(x)w \quad \text{and} \quad [Gu](x) \triangleq g(x)u
\] (23)
are all bounded linear operators with finite rank.
Let \( z(t) = [\theta(t, \cdot) \ w(t)]^T \in L^2(0, L) \times \mathbb{R}^n \), then the hybrid system (21) - (22) can be written as the distributed parameter system on the state space \( Z = X \times \mathbb{R}^n = L^2(0, L) \times \mathbb{R}^n \) as
\[
\dot{z}(t) = \mathcal{A}z(t) + \mathcal{B}u_\epsilon(t) \tag{24}
\]
where \( D(\mathcal{A}) = D(\mathcal{A}_0) \times \mathbb{R}^n \leq L^2(0, L) \times \mathbb{R}^n \),
\[
\mathcal{A} = \begin{bmatrix} \mathcal{A}_0 & F \\ 0 & A \end{bmatrix}
\]
and the input operator has the form
\[
\mathcal{B} = \begin{bmatrix} G \\ B \end{bmatrix}
\]
Here, \( F \) and \( G \) are defined in (23) and the important point is that \( \mathcal{B} : \mathbb{R}^m \to Z = L^2(0, L) \times \mathbb{R}^n \) is a bounded linear operators. Moreover, we have the following result.

**Theorem I:** The linear operator \( \mathcal{A} \) generates a \( C_0 \)-semigroup \( S(t) \) on \( Z = L^2(0, L) \times \mathbb{R}^n \). Moreover, if \( \kappa > 0 \) and \( A \) is stable, then the composite semigroup is exponentially stable and the controlled system (24) is stabilizable.

**Outline of Proof:** As noted above, if \( \kappa > 0 \), then \( \mathcal{A}_0 \) generates an exponentially stable nilpotent semigroup \( S_0(t) \) on \( L^2(0, L) \), \( A \) generates an exponentially stable semigroup \( e^{\lambda t} \) on \( \mathbb{R}^n \) and
\[
\mathcal{A} = \begin{bmatrix} \mathcal{A}_0 & F \\ 0 & A \end{bmatrix}
\]
where \( F : \mathbb{R}^n \to L^2(0, L) \) is defined by (23) is bounded. Standard perturbation theorems (e.g., Theorem 1.1 on page 76 in [27]) imply that the operator \( \mathcal{A} \) generates a \( C_0 \)-semigroup \( S(t) \) on \( X \).

First note that \( \rho(\mathcal{A}_0) \) is the entire complex plane so that the spectrum of \( \mathcal{A}_0 \) is empty and for all \( \lambda \), the resolvent \( [\lambda I - \mathcal{A}_0]^{-1} \) exists. Consider the eigenvalue problem
\[
\mathcal{A} \begin{bmatrix} \phi(\cdot) \\ w \end{bmatrix} = \begin{bmatrix} \mathcal{A}_0 & F \\ 0 & A \end{bmatrix} \begin{bmatrix} \phi(\cdot) \\ w \end{bmatrix} = \lambda \begin{bmatrix} \phi(\cdot) \\ w \end{bmatrix}.
\]
The first equation implies that
\[
\mathcal{A}_0 \phi(\cdot) + [Fw](\cdot) = \lambda \phi(\cdot)
\]
and the second equation is
\[
Aw = \lambda w.
\]
If \( w_\lambda \neq 0 \) is an eigenvector for \( A \) corresponding to \( \lambda \in \sigma_p(A) \), then \( [\lambda - \mathcal{A}_0]^{-1} \phi_\lambda = Fw_\lambda \) has a unique solution
\[
\phi_\lambda(\cdot) = [\lambda I - \mathcal{A}_0]^{-1} Fw_\lambda,
\]
and hence \( z_\lambda = [\phi_\lambda(\cdot) \ w_\lambda]^T \in D(\mathcal{A}) \) is an eigenvector of \( \mathcal{A} \). On the other hand, if \( \lambda \in \rho(\mathcal{A}) \), then \( \lambda I - A \) is surjective and the only solution to \( Aw = \lambda w \) is \( w_0 = 0 \). In this case the equation
\[
\mathcal{A}_0 \phi(\cdot) + [Fw_\lambda](\cdot) = \lambda \phi(\cdot)
\]
reduces to
\[
\mathcal{A}_0 \phi(\cdot) = \lambda \phi(\cdot)
\]
which implies \( \phi(\cdot) = 0 \). It follows that \( \sigma_p(\mathcal{A}) = \sigma_p(A) \) and if \( \lambda \in \sigma_p(\mathcal{A}) \), then
\[
z_\lambda = [\lambda I - \mathcal{A}_0]^{-1} Fw_\lambda \ w_\lambda^T
\]
is the eigenvector corresponding to \( \lambda \in \sigma_p(\mathcal{A}) \). Since the composite operator \( \mathcal{A} \) is a spectral operator, it generates an exponentially stable \( C_0 \)-semigroup \( S(t) \) on \( Z \) and this completes the proof. \( \square \)

### III. The LQR Control Problem

Assume one begins with an LQR problem defined by the boundary control problem (4)-(5) with cost function
\[
J = \int_0^\infty \left\{ \langle QT(t, \cdot), T(t, \cdot) \rangle_{L^2} + \langle Ru_0(t), u_0(t) \rangle \right\} dt \tag{27}
\]
where \( Q = Q^* \geq 0 \) is a bounded linear operator on \( L^2(0, L) \) and \( R > 0 \). If one assume that \( u_0(t) = Cw(t) \) is the output to the actuator equations (18), then
\[
J = \int_0^\infty \left\{ \langle QT(t, \cdot), T(t, \cdot) \rangle_{L^2} + \langle RCw(t), Cw(t) \rangle \right\} dt
= \int_0^\infty \left\{ \langle QT(t, \cdot), T(t, \cdot) \rangle_{L^2} + \langle Q_u w(t), w(t) \rangle \right\} dt,
\]
where \( Q_u = C^T RC \geq 0 \). Therefore, we consider the LQR problem for the composite system (25) given by
\[
\dot{z}(t) = \mathcal{A}z(t) + \mathcal{B}u_\epsilon(t) \tag{28}
\]
with cost
\[
J_c = \int_0^\infty \left\{ \langle \mathcal{D}z(t), z(t) \rangle_{L^2} \right\} dt \tag{29}
+ \int_0^\infty \left\{ \langle R_u u_\epsilon(t), u_\epsilon(t) \rangle_{\mathbb{R}^m} \right\} dt
\]
where \( \mathcal{D} \) has the form
\[
\mathcal{D} = \begin{bmatrix} Q_{1,1} & Q_{1,2} \\ Q_{2,1} & Q_{2,2} \end{bmatrix}
\]
and \( R_c > 0 \) is a weighting matrix on the actuator control. Define \( E : \mathbb{R}^n \to L^2(0, L) \) by \( [Ew](x) = b(x)Cw \) which implies that \( Q_{1,2} = QE \) and \( Q_{2,2} = E^*QE + Q_u \) where \( E^* : L^2(0, L) \to \mathbb{R}^n \) is given by
\[
E^* \phi(\cdot) = C^T \int_0^L b(\xi)\phi(\xi) d\xi.
\]
The details of this transformation are given in [10] and will not be presented here in order to save space.

We assume the actuator system is stable so that \( \mathcal{A} \) generates an exponentially stable \( C_0 \)-semigroup \( T(t) \) on \( Z = L^2(0, L) \times \mathbb{R}^n \), and \( R_c = [R_c]^T > 0 \). Thus, (since \( \mathcal{D} \) is bounded and nonnegative definite) there exist a unique optimal control \( u^{opt}_{\epsilon}(t) \) to the LQR problem defined by (28)-(29) and
\[
u^{opt}_{\epsilon}(t) = - \mathcal{H} \ z^{opt}_{\epsilon}(t).
\]
Moreover, the Riesz representation theorem implies that \( \mathcal{H} : Z = L^2(0, L) \times \mathbb{R}^n \to \mathbb{R}^m \) has the form
\[
\mathcal{H} \begin{bmatrix} \phi(\cdot) \\ w \end{bmatrix} = \int_0^L k(x)\phi(x) dx + \hat{K} w.
\]

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Here, \( k(x) = [k_1(x) \ k_2(x) \ \ldots \ k_m(x)]^T \) with \( k_i(\cdot) \in L^2(0,L), \ i = 1,2,\ldots,m \) and \( \tilde{k}^c \) is an \( m \times n \) matrix \( i = 1,2,\ldots,m, \ j = 1,2,\ldots,n \). The functions \( k_i(\cdot) \) are called the functional gains. Since \( z(t) = [\theta(t,\cdot) \ w(t)]^T \) the feedback controller has the form

\[
\mathbf{u}^F(t) = -\int_0^L k(x)\theta(t,\xi)d\xi - \tilde{k}^c\mathbf{w}(t). \tag{31}
\]

Thus, in terms of the original states we have

\[
\dot{\mathbf{z}}^F(t) = -\int_0^L k(\xi)\mathbf{T}(t,\xi)d\xi - \tilde{k}^c\mathbf{w}(t),
\]

where

\[
\tilde{k}^c = \frac{1}{h} - \int_0^L k(x)b(x)Cd\chi.
\]

Consequently, once one computes the functional gains \( k(\cdot) \) and the fixed gain \( \tilde{k}^c \), computing the gain matrix \( \tilde{k}^c \) requires only a quadrature.

IV. NUMERICAL APPROXIMATIONS OF THE COMPOSITE SYSTEM

We apply a finite volume scheme which is similar to the “AVE scheme” developed for delay systems in [1] and [3]. The basic idea is to approximate the initial conditions \( \phi(\cdot) \in L^2(0,L) \) by step functions with values equal to the average of the function on the subintervals. Let \( N > 1 \) and create a partition on [0, L] by defining \( \chi^0_j = jh \), where \( h = L/N \) and \( j = 0,\ldots,N \). Define \( \chi^N_j : [0,L] \to \mathbb{R} \) to be the characteristic functions on \( [\chi^N_{j-1}, \chi^N_j) \), for \( j = 1,2,\ldots,N-1 \) and \( \chi^N_{N-1} \) to be the characteristic function on \( [0,\chi^N_{N-1},L] \). Let \( \Phi_N \) be the closed subspace of \( L^2(0,L) \) defined by

\[
\Phi_N = \left\{ \phi(x) = \sum_{j=1}^N c_j^N \chi^N_j(x), \quad c_j^N \in \mathbb{R} \right\}. \tag{32}
\]

The orthogonal projection \( \Pi_N \) of \( L^2(0,L) \) onto \( \Phi_N \) is defined by

\[
\Pi_N \phi(\cdot) = \sum_{j=1}^N \phi_j^N \chi_j^N(\cdot), \tag{33}
\]

where for \( j = 1,\ldots,N \),

\[
\phi_j^N = \frac{N}{L} \int_{\chi^N_{j-1}}^{\chi^N_j} \phi(x)dx = \frac{1}{h} \int_{\chi^N_{j-1}}^{\chi^N_j} \phi(x)dx. \tag{34}
\]

are the mean values. Approximating the differential operator \( \partial_0 \) is now straightforward. Define \( \mathcal{D}_0 : \Phi_N \to \Phi_N \subseteq L^2(0,L) \) by

\[
\mathcal{D}_0 \phi(x) = -\frac{\nu}{h} \sum_{j=1}^N [\phi_j^N - \phi_{j-1}^N] \chi_j^N(x) + -\kappa \sum_{j=1}^N \phi_j^N \chi_j^N(x), \tag{35}
\]

where we set \( \phi_0^N = 0 \).

Recall that the operators \( F : \mathbb{R}^n \to L^2(0,L) \), \( G : \mathbb{R}^m \to L^2(0,L) \) and \( E : \mathbb{R}^n \to L^2(0,L) \) are defined by \( [Fw](x) = -b(x)Ca w \), \( [Gu](x) = -b(x)Cb u \) and \( [Ew](x) = b(x)Cw \), respectively. Thus, all these operators can be approximated by projecting the function \( b(\cdot) \in L^2(0,L) \) onto \( \Phi_N \). A direct computation yields

\[
b^N(x) = \sum_{j=1}^N b_j^N \chi^N_j(x)
\]

where

\[
b_j^N = \left\{ \begin{array}{ll}
1, & \kappa = 0 \\
\frac{-u}{\kappa} [e^\frac{-\kappa h}{u} - e^\frac{u}{\kappa}h], & j = 1,2,\ldots,N, \kappa > 0
\end{array} \right.
\]

Thus, one obtains the approximating operators

\[
[F^Nw](x) = [-b^N(x)Ca w],
\]

\[
[G^Nw](x) = [-b^N(x)Cb u],
\]

\[
[E^Nw](x) = [b^N(x)Cw],
\]

Define \( Z_N \subseteq Z = L^2(0,L) \times \mathbb{R}^n \) to be the subspace

\[
Z_N = \Phi_N \times \mathbb{R}^n
\]

and let \( \mathcal{A}_N : Z_N \to Z_N \) be defined by

\[
\mathcal{A}_N = \begin{bmatrix} \mathcal{D}_0^N & F^N \\ 0 & A \end{bmatrix}.
\]

The approximating input operator \( \mathcal{B}_N : \mathbb{R}^m \to Z_N \) is given by

\[
\mathcal{B}_N = \begin{bmatrix} G^N \\ B \end{bmatrix}
\]

and hence one arrives at the approximating control system

\[
\dot{\mathbf{z}}^N(t) = \mathcal{A}_N \mathbf{z}^N(t) + \mathcal{B}_N \mathbf{u}_c(t). \tag{39}
\]

Simple calculations produce the matrix representation of this system as

\[
\dot{\mathbf{z}}^N(t) = A^N \mathbf{z}^N(t) + B^N \mathbf{u}_c(t) \tag{40}
\]

where

\[
A^N = \begin{bmatrix} D^N & F^N \\ 0 & A \end{bmatrix},
\]

\[
D^N = D_1^N + D_2^N,
\]

and

\[
D_1^N = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ -1 & 1 & 0 & \cdots & 0 \\ 0 & -1 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & -1 & 1 \end{bmatrix},
\]

and

\[
D_2^N = \frac{-\kappa}{h} \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix} = \frac{-\kappa}{h} I.
\]
respectively. Also,

\[ F^N = - \begin{bmatrix} b_1^N & b_2^N & \cdots & b_N^N \end{bmatrix} \]

and the control matrix is given by

\[ B^N = \begin{bmatrix} G^N \\ B \end{bmatrix}, \]

where

\[ G^N = - \begin{bmatrix} b_1^N & b_2^N & \cdots & b_N^N \end{bmatrix} \]

A modification of the proof in [1] can be used to establish the following result.

**Theorem 2:** Let \( S^N(t) \) denote the semigroup generated by the linear operator \( \mathcal{A}^N \) and for each \( z = [\phi(\cdot), w(t)]^T \in Z = L^2(0,L) \times \mathbb{R}^n \) set \( z^N = [\Pi^N \phi(\cdot), w(t)]^T \in Z^N \). If \( \kappa > 0 \), then

\[ \| S^N(t) z - S(t) z \| \to 0 \]

as \( N \to +\infty \) and the convergence is uniform on compact time intervals. Moreover, the operators \( \mathcal{B}^N \) converge in operator norm to \( \mathcal{B} \).

Consider the approximating LQR problem defined by

\[ J_c^N = \int_0^{+\infty} \left\{ \langle \mathcal{A}^N \mathcal{Z}^N(t), \mathcal{Z}^N(t) \rangle_{Z^N} \right\} dt + \int_0^{+\infty} \left\{ \langle \mathcal{R}_c \mathcal{U}_c(t), \mathcal{U}_c(t) \rangle_{R^m} \right\} dt. \]

Assuming convergence of the dual system (see [7], [8] and [9]) one can show that the approximating feedback gains converge strongly to the infinite dimensional functional gains (see [16]). In particular,

\[ \| k_1^N(\cdot) - k(\cdot) \|_{L^2} \to 0. \]

It is possible to establish dual convergence for this scheme, but the proof is too long to be given in this short paper. However, we illustrate this convergence with a simple numerical example.

### V. Numerical Example

We consider the 1D control problem for the system (4)-(5) with actuator dynamics defined by the scalar system

\[ \dot{w}(t) = aw(t) + bu_0(t) \quad (41) \]

and output

\[ u_0(t) = cw(t), \]

where \( L = 1, \ \nu = 0.1, \ \kappa = 1.0, \ a = -5, \ b = 5 \) and \( c = 1 \). The state weighting operator is the identity on \( L^2(0,L) \) and the control weights are \( R = 1 \) and \( R_c = 0.1 \), respectively. For this case the optimal control has the form

\[ u_0^{opt}(t) = - \int_0^L k(\xi) T(t,\xi)d\xi - k^c w(t) \]

and the goal is to compute the approximations \( k^N(\cdot) \) of \( k(\cdot) \). Note also, that the approximating system produces approximate values \( [k^c]^N \) for the constant gains \( k^c \).

Figure 1 illustrates the convergence of the functional gains \( k^N(\cdot) \). Shown are plots of \( k^N(\cdot) \) for \( N = 256, 512, 1024 \). Observe that to capture the peak of the functional gain, the mesh had to be refined considerably. However, convergence away from \( x = 0 \) is rather rapid and this suggests a non-uniform mesh should be employed. We plan to investigate this issue in a full paper.

It is interesting to note that the constant gains \( [k^c]^N \) are the same at each level of approximation, i.e. \( [k^c]^N = 2.3166 \) for all \( N > 4 \) so the only error in \( k^c \) is due to the integral term in

\[ k^c = \tilde{k}^c - \int_0^L k(x)b(x)dx. \]

The table below contains values of \( [k^c]^N \).

<table>
<thead>
<tr>
<th>N</th>
<th>( [k^c]^N )</th>
</tr>
</thead>
<tbody>
<tr>
<td>32</td>
<td>2.0231</td>
</tr>
<tr>
<td>64</td>
<td>2.0284</td>
</tr>
<tr>
<td>128</td>
<td>2.0333</td>
</tr>
<tr>
<td>256</td>
<td>2.0331</td>
</tr>
<tr>
<td>512</td>
<td>2.0331</td>
</tr>
</tbody>
</table>

Observe that the constant gains converge rapidly since the function \( b(\cdot) \) decreases rapidly on \( (0,1) \).

### VI. Conclusions

Boundary control of hyperbolic systems of the form (4)-(5) do not fall into the classes of boundary control problems treated in the existing literature. In order to deal with the issues 1)-4) above, we included the effects of adding actuator dynamics. This allowed us to apply finite volume type
methods to a control problem with a compact input operator. Moreover, the inclusion of actuators offers a more realistic model of the dynamics of the coupled system and allows for the integration of system components. The numerical example illustrated the convergence of the control system. However, to provide a complete theoretical basis for this convergence one needs to establish the dual convergence. This will appear in a future paper. In the paper [10] we considered the case where the system was parabolic with actuator dynamics. Delayed actuator dynamics have also been considered for these analytic systems (see [11] and [12]).

VII. ACKNOWLEDGMENTS

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REFERENCES


