On the Robustness of Event-Based Synchronization under Switching Interactions

Hamed Shisheh Foroush and Sonia Martínez

Abstract—In this paper we study the robustness of an event-triggered synchronization dynamics for a network of identical nodes under various switching scenarios. We first consider an arbitrary switching scenario where, for a general class of isolated node dynamics we characterize sufficient conditions in terms of network topologies to maintain synchronization. In particular, we shall also demonstrate that for a specific class of skew-symmetric isolated node dynamics—which play important role in this class of synchronization problems—the asymptotic synchronization is not achievable under arbitrary switching. We then consider two classes of constrained switching signals, namely uniform and average classes, i.e., $S_{\text{uniform}}[TD]$, and $S_{\text{average}}[\tau_0, N_0]$, respectively, where we characterize sufficient conditions in terms of the associated parameters, $TD$, $\tau_0$ and $N_0$ in order to ensure asymptotic synchronization. We shall wrap up our discussion by presenting relevant simulation studies.

I. INTRODUCTION

Cyber-Physical Systems (CPS) are physical plants which are remotely controlled and monitored via wireless or wired communication channels. Due to the widespread deployment of cps systems in general infrastructure systems, they have gathered significant attention in the past few years. More specifically, amongst various examples of CPS, one may count general example systems such as the smart power grid. In nontechnical words, a smart grid entails a number of power generators communicating with each other to produce and supply electric energy to a network of consumers. In more technical words, the dynamics governing the smart grid application is cast under synchronization dynamics [1], [2].

The aforementioned synchronization dynamics has been studied mainly under two categories: (i) with identical node (oscillator) dynamics and (ii) with different node (oscillator) dynamics. The major review on synchronization [2] discusses the differences and resemblances between these two classes; in this paper we shall focus on the the first class of synchronization dynamics, i.e., with identical node dynamics. We note that, as discussed in [1], this class encompasses the dynamics representing a smart grid application—which further motivates the present study.

There exists already a substantial literature within the controls community dedicated to study this specific class of synchronization dynamics. To mention a few, in [3], the authors study the stability of this type of dynamics by introducing a so-called master stability function which characterizes the maximum Lyapunov exponent of the governing dynamics. The papers [4], [5], [6] study this problem in the context of switched systems, where switching amongst different potential network topologies has been considered and, thus, by characterizing the dynamics of the error variable between the network state and the average state, some switching rules have been derived in order to achieve network synchronization. In these latter studies, it is nonetheless worth noting that the communication is performed in a continuous fashion. In [7], [8] the authors study the synchronization problem in the context of event-triggered dynamics for a fixed network topology, while proposing centralized and distributed event-triggered rules, respectively, to ensure network synchronization. In order to do so, a set-stability technique is exploited.

Hence, there is an apparent gap in the literature in terms of studying synchronization dynamics by considering event-triggered communications and in the context of switched systems. This work aims to close this gap by characterizing sufficient conditions on switched networked topologies and robustness conditions on switching signals that can ensure network synchronization. Indeed, the CPS nature of the smart grid application also motivates this study by economic number of communications governed by considering event-triggered dynamics and the potential unavailability of power generators in a smart grid.

II. PRELIMINARIES AND PROBLEM FORMULATION

In this section, we first present some notations and preliminaries on the synchronization, event-based synchronization and switching concepts. This is then followed by the problems that we study in this paper.

We consider a network of $N$ identical oscillators, with $m$ number of possible topologies. Let us consider $k \in \{1, \ldots, m\}$ to be the $k^{th}$ topology of the network, and $x_i \in \mathbb{R}^n$, for $i \in \{1, \ldots, N\}$, be the state of the $i^{th}$ node.
Then, the dynamics of this node is as follows:
\[
\dot{x}_i = H x_i + c \sum_{j=1}^{N} a_{ij}^{k} \Gamma x_j (t_p^{k}), \quad \forall t \in [t_p^{k}, t_{p+1}^{k}],
\]
wherein \( H \in \mathbb{R}^{n \times n} \) states the dynamics of each node, \( c > 0 \) is the coupling strength, and \( \{t_p^{k}\} \) is the triggering time-sequence associated to the \( k \)-th topology. Also, \( \Gamma \in \mathbb{R}^{n \times n} \) is the inner-coupling matrix, and \( A^k = [a_{ij}^{k}] \in \mathbb{R}^{N \times N} \) is the outer-coupling matrix for the \( k \)-th topology. We further assume that each network is undirected, connected, and balanced, which can then be induced that matrices \( A^k \) are symmetric, irreducible, and with the zero-sum property, where the last property implies \( a_{ii}^k = \sum_{j=1, j \neq i}^{N} a_{ij}^k = - \sum_{j=1, j \neq i}^{N} a_{ji}^k \) or in other words, \( 1_N^\top A = A 1_N = 0 \) for \( i \in \{1, 2, \ldots, N\} \). Indeed, it can also be observed that in this context, outer-coupling matrix, \( A^k \), plays the role of the negative of Laplacian matrix.

Let also \( \{\lambda_i^{k}\}_{i \in \{1, \ldots, N\}} \) be the set of eigenvalues of the \( k \)-th topology. Given the properties of the \( A^k \) matrix, we realize that these eigenvalues are real and that \( \lambda_1^{k} = 0 > \lambda_2^{k} \geq \lambda_3^{k} \geq \cdots \geq \lambda_N^{k} \) holds. In addition, let \( \{\phi_i^{k}\}_{i \in \{1, \ldots, N\}} \), where \( \phi_i^{k} \in \mathbb{R}^N \), be the correspondent set of eigenvectors. In addition, let \( I_N \) and \( 1_N \) be, respectively, the identity and matrix of all ones with dimension, \( N \). At last, we would like to denote by, \( |x|_{A} \), the Euclidean point-to-set distance between vector, \( x \), and manifold, \( A \), defined as \( | x |_A = d (x, A) = \inf_{y \in A} \| x - y \| , \) with \( \| \cdot \| \) is the Euclidean norm.

Then, the following result holds.

**Lemma 2.1:** The following properties hold:

1. \( \{\phi_i^{k}\}_{i \in \{1, \ldots, N\}} \) can be selected to be an orthonormal set of eigenvectors, with \( \phi_i^{k} = \frac{1}{\sqrt{N}} [1, 1, \ldots, 1]^\top \) associated to \( \lambda_1^{k} = 0 \) and \( \{\phi_2^{k}, \ldots, \phi_N^{k}\} \) be such that \( \sum_{j=1}^{N} \phi_j^{k} = 0 \), for \( i \in \{2, \ldots, N\} \).

2. Let \( \Phi^k = [\phi_1^{k}, \ldots, \phi_N^{k}] \in \mathbb{R}^{N \times N-1} \) then \( \Phi^k \Phi^k \top = I_{N-1} \) and \( \Phi^k \Phi^k \top = I_N - \frac{1}{N} 1_N \top 1_N \).

3. Matrix \( A^k \) is diagonalizable with \( \Phi^k \top A^k \Phi^k = \Lambda^k = \text{diag} \{\lambda_2^{k}, \ldots, \lambda_N^{k}\} \in \mathbb{R}^{N-1} \).

We then incorporate the notion of Kronecker product, \( \otimes \), in order to obtain the network dynamics:
\[
\dot{x}(t) = (I_N \otimes H) x(t) + c (A^k \otimes \Gamma) x(t_p^{k}), \quad \forall t \in [t_p^{k}, t_{p+1}^{k}],
\]
where we recall \( \{t_p^{k}\}_{p \in \mathbb{N}} \) is the triggering time-sequence associated to the \( k \)-th network topology. We then define the switching signal, \( \sigma : \mathbb{R}_{\geq 0} \to \{1, 2, \ldots, m\} \), in order to declare network topology at the time-instant, \( t \). We also assume that the switching signal, \( \sigma(t) \), is piecewise constant. In addition, we assume that \( \sigma(t) \) is continuous from above, i.e., \( \forall t \geq 0, \lim_{s \downarrow t} \sigma(s) = \sigma(t) \). In this paper, we shall denote the switching time-instants—which are indeed the discontinuities of \( \sigma(t) \)—to be sequentially \( \tau_j \) for \( j \in \mathbb{N}_0 \). Hereby and resorting to dynamics (1), we would like to clarify that upon every switching occurrence, the communication pattern, i.e., the topology based on which triggering time-sequence is derived, also switches. Therefore, accordingly, there is only one network topology which switches from time to time and according to which synchronization is performed and the triggering time-sequence is derived.

We next define synchronization in the following formal way. Let first \( x(t; x_0) = (x_1(t; x_0)^\top, x_2(t; x_0)^\top, \ldots, x_N(t; x_0)^\top)^\top \in \mathbb{R}^N \) be a solution of the network dynamics (1) with initial condition \( x_0 = (x_1(0)^\top, x_2(0)^\top, \ldots, x_N(0)^\top)^\top \) and some triggering time-sequence \( \{t_p^{k}\} \) and switching signal \( \sigma(t) \), the synchronization is then defined as follows that is along the lines of [7].

**Definition 2.2:** Let \( A_s = \{x \in \mathbb{R}^N | x_1 = x_2 = \cdots = x_N \} \), with \( x = (x_1^\top, x_2^\top, \ldots, x_N^\top)^\top \), be the synchronization manifold. If then there exists a \( \delta > 0 \) such that the limit \( \lim_{t \to \infty} |x(t; x_0)|_{A_s} = 0 \), holds whenever \( |x_0|_{A_s} < \delta \), then the network (1) is said to achieve local asymptotic synchronization. Moreover, if \( \delta = \infty \), then global asymptotic synchronization is achieved.

We would also like to recall two switching scenarios: (i) arbitrary, and (ii) constrained, where respectively, the switching signal, \( \sigma(t) \) does not and does obey a specific temporal structure. In this paper, we consider two classes of constrained switching signals; namely, \( S_{\text{dwell}}[\tau_D] \), with \( \tau_D > 0 \), and \( S_{\text{average}}[\tau_a, N_0] \), with \( \tau_a, N_0 > 0 \). The class, \( S_{\text{dwell}}[\tau_D] \), constitutes of switching signals \( \sigma(t) \) where any two consecutive discontinuities of \( \sigma \) are separated by at least a dwell time, \( \tau_D \), therefore, \( \tau_j+1 - \tau_j \geq \tau_D, \forall j \in \mathbb{N}_0 \). Moreover, the class, \( S_{\text{average}}[\tau_a, N_0] \), contains the switching signals \( \sigma(t) \) for which the following holds:
\[
N_s(\tau, t) \leq N_0 + \frac{\tau - t}{\tau_a}, \quad \forall t \geq \tau \geq 0,
\]
where \( N_s(\tau, t) \) denotes the number of discontinuities of \( \sigma \) in the open interval, \((\tau, t)\); and the constant \( \tau_a \) is called the average dwell-time and \( N_0 \) is the chatter bound.

At last, the problems we study in this paper can be stated as follows. Given the network dynamics (1) and in order to achieve asymptotic synchronization as characterized in Definition 2.2, which triggering strategy to be employed in order to generate the triggering time-sequence \( \{t_p^{k}\} \); and in addition:

1. **Problem 1:** considering arbitrary switching scenario, which sufficient conditions have to be imposed on the network topologies.
2. **Problem 2:** considering constrained switching scenario and under \( S_{\text{dwell}} \) and \( S_{\text{average}} \) classes, which sufficient regulatory conditions have to be imposed on the switching signals.
III. SINGLE TOPOLOGY: LYAPUNOV FUNCTION AND TRIGGERING STRATEGY CHARACTERIZATION

In this section, we shall focus on a single topology scenario whereby we first characterize our specific Lyapunov function along with the triggering strategy. The content of this section is inspired from the results in [7], nonetheless we propose alternative expanded proofs, which can be found in [9].

We shall first, without loss of generality, drop the superscript $k$ for various variables in this section. Let us then recall the matrix of eigenvectors of $A$, i.e., $Φ$, as described and characterized in Lemma 2.1. We then introduce $Φ = Φ \otimes I_n \in \mathbb{R}^{nN \times n(N-1)}$ whereby we get the following result.

Lemma 3.1: Consider network (1) and the network state, $x$, then the following holds:

$$\|Φ^\top x\| = |x|_{A_n}.$$

Remark 3.2: Indeed, motivated by the previous result, one can define $y = Φ^\top x$ to be the component of state vector, $x$, which evolves traverse to the synchronization manifold. Therefore, resorting to Definition 2.2, by ensuring $\lim_{t \to \infty} \|y(t; y_0)\| = 0$, we shall ensure $\lim_{t \to \infty} |x(t; x_0)|_{A_n} = 0$, and thus we can ensure the asymptotic synchronization of the original state, $x$.

In the next result and prior to discussing the triggering strategy to be analyzed, we introduce a proper Lyapunov function to ensure synchronization of the network:

$$\dot{x}_i = Hx_i + c \sum_{j=1}^{N} a_{ij}Γx_j(t) \quad i \in \{1, 2, \ldots, N\}. \quad (2)$$

Proposition 3.3: If there exist matrices $P_i = P_i^\top > 0 \in \mathbb{R}^{n \times n}$ such that

$$H_i^\top P_i + P_i H_i < 0, \quad i \in \{2, 3, \ldots, N\},$$

with $H_i = H + c\lambda_iΓ$, then

$$V(x) = x^\top ΦPΦ^\top x,$$

is a Lyapunov function for the network (2), i.e., it ensures its asymptotic synchronization, with $P = \text{diag}\{P_2, P_3, \ldots, P_N\}$ and $\lambda_i$ be eigenvalues of $A$ and $Φ = Φ \otimes I_n$, as discussed in Lemma 3.1.

Having discussed and proven a proper Lyapunov function in the previous proposition, let us recall now the triggered dynamics (1) (for the case of single topology), which by introducing the error variable, $e(t) = x(t) - x(t)$, can be reformulated as follows:

$$\dot{e}(t) = (I_N \otimes H + cA \otimes Γ)e(t) + cA \otimes Γe(t), \quad \forall t \in [t_p, t_{p+1}]. \quad (3)$$

The following result characterizes an event-triggering strategy addressing how to generate the time-sequence, $\{t_p\}_{p\in \mathbb{N}}$, whilst (i) maintaining the asymptotic synchronization of the network and, (ii) avoiding Zeno behavior.

Proposition 3.4: Consider the triggered network dynamics (3), and assume there exist matrices $P_i = P_i^\top > 0 \in \mathbb{R}^{n \times n}$ such that

$$H_i^\top P_i + P_i H_i = -2I_n, \quad i \in \{2, 3, \ldots, N\}, \quad (4)$$

with $H_i = H + c\lambda_iΓ$, where $\lambda_i$ be eigenvalues of $A$ and $Φ = Φ \otimes I_n$. Then, the asymptotic synchronization is ensured under the following event-triggered strategy:

$$t_{p+1} = \inf \left\{ t > t_p \mid \|Φ^\top e(t)\| \geq \frac{\delta}{\alpha}\|Φ^\top x(t)\| \right\}, \quad p \in \mathbb{N},$$

with $e(t) = x(t_p) - x(t)$, $\delta \in (0, 1)$, and, $\alpha = \max_{i \in \{2, 3, \ldots, N\}} \{-c\lambda_i\|P_i Γ\|\}$. In addition, under this triggering strategy, the Lyapunov function $V(x) = x^\top ΦPΦ^\top x$, with $P = \text{diag}_{i \in \{2, 3, \ldots, N\}}\{P_i\}$, satisfies the following dynamics:

$$\dot{V}(x) < -2(1 - \delta)\|Φ^\top x\|^2, \quad \forall t \in [t_p, t_{p+1}].$$

In addition, $\exists 0 > r$ such that $\forall p \in \mathbb{N}$, $t_{p+1} - t_p \leq r.$

Remark 3.5: We would like to resort to the triggering rule (5) introduced in Proposition 3.4. We note that it is a centralized triggering rule, i.e., all the nodes of the network trigger at the same time. Admittedly, instead, this could have been a decentralized triggering rule; however, we have left it for a future work study. Note that the decentralized triggering rule developed in [7] (i) does not (provably) exclude Zeno behavior and that (ii) it requires that every node knows the Fielder eigenvalue of the network which is, in essence, a global knowledge requested at every node and thus a strong assumption.

IV. MULTIPLE TOPOLOGIES: ARBITRARY SWITCHING

In this section, we shall address problem 1 stated and discussed in Section II. The analysis of this section is performed under two cases: (i) General Network Dynamics, wherein matrix $H$ in dynamics (1) can be any given matrix, and (ii) Particular Network Dynamics, wherein matrix $H$ is restricted to be a skew-symmetric matrix.

a) General Network Dynamics: Let us recall from [10] that in order to ensure the uniform asymptotic stability of a switched system, $\dot{x} = f_p(x)$, with $x \in \mathbb{R}^n$, for the case of arbitrary switching, and for a family of topologies, $p \in \mathcal{P}$, it is sufficient to find a common Lyapunov function, i.e., a positive-definite continuously differentiable function, $V : \mathbb{R}^n \to \mathbb{R}$, and a positive-definite continuous function, $W : \mathbb{R}^n \to \mathbb{R}$, such that the following holds:

$$\dot{V}(x) = \frac{\partial V}{\partial x} f_p(x) \leq -W(x), \quad \forall x \in \mathbb{R}^n, \quad \forall p \in \mathcal{P}.$$
Hence, in order to adopt this latter argument to the case of switched triggered synchronization dynamics (1), we have opted for finding a Lyapunov function of the earlier-discussed form: $V^k(x) = x^T P^k \dot{V}^k x$ which is common for every topology, $k \in \{1, \ldots, m\}$. This latter is because this specific class of Lyapunov functions has been used to derive the triggering strategy we analyze in our study. The following result studies the existence of such a Lyapunov function for specific network topologies.

**Proposition 4.1:** Consider dynamics (1), where $k \in \{1, \ldots, m\}$, and assume there exist matrices $P^k_i = P^k_{i \rightarrow 0} \in \mathbb{R}^{n \times n}$, $i \in \{2, 3, \ldots, N\}$, satisfying condition (4), for every $k \in \{1, \ldots, m\}$. Moreover, let $\delta \in (0, 1)$, $e(t) = x(t^k_i) - x(t)$, $\alpha_k = \max_{i \in \{2, 3, \ldots, N\}} \{ -\delta \lambda^2 P^k_i \}$, and $\tau_D = \max_{i \in \{2, 3, \ldots, N\}} \{ P^k_i \}$, and $\{ t^k_i \}$ be the triggering time-sequence generated by (5), that is restated as follows:

$$\| \dot{V}^k e(t^k_i) \| = \frac{\delta}{\alpha_k} \| \dot{V}^k e x(t^k_i) \|, p \in \mathbb{N}, k \in \{1, \ldots, m\}. $$

Then, the asymptotic synchronization of (1) is ensured under arbitrary switching scenario if:

$$\dot{V}^k_i (\text{dia}_{i \in \{2, \ldots, N\}} (H^T + H + c \lambda^2 (H^T + H + c \lambda^2)), k \in \{1, \ldots, m\}. $$

**b) Particular Network Dynamics:** Motivated by our results stated and discussed in previous paragraph, in this paragraph, we shall narrow our studies down to a particular class of dynamics, for which $H + H^T = 0$, i.e., matrix $H \in \mathbb{R}^{n \times n}$—which characterizes the dynamics of each node—be skew-symmetric. We note that this class of dynamics is interesting provided our discussion in Appendix A of the complete version of this paper which can be found in [9]. Having stated these points and along the lines of Proposition 4.1, we shall establish the following proposition for this particular class of dynamics.

**Proposition 4.2:** Consider dynamics (1), where $k \in \{1, \ldots, m\}$, and assume first that the matrix $H$ is skew-symmetric, and second that there exist matrices $P^k_i = P^k_{i \rightarrow 0} \in \mathbb{R}^{n \times n}$, $i \in \{2, 3, \ldots, N\}$, satisfying condition (4), for every $k \in \{1, \ldots, m\}$. Moreover, let $\delta \in (0, 1)$, $e(t) = x(t^k_i) - x(t)$, $\alpha_k = \max_{i \in \{2, 3, \ldots, N\}} \{ -\delta \lambda^2 P^k_i \}$, and $\tau_D = \max_{i \in \{2, 3, \ldots, N\}} \{ P^k_i \}$, and $\{ t^k_i \}$ be the triggering time-sequence generated by (5), that is restated as follows:

$$\| \dot{V}^k e(t^k_i) \| = \frac{\delta}{\alpha_k} \| \dot{V}^k e x(t^k_i) \|, p \in \mathbb{N}, k \in \{1, \ldots, m\}. $$

Then, the asymptotic synchronization of (1) can never be ensured under arbitrary switching scenario.

**V. MULTIPLE TOPOLOGIES: SWITCHING SIGNAL DESIGN**

In this section we shall recall two classes of switching signals introduced in Section II, i.e., $S_{\text{dwell}}[\tau_D]$ and $S_{\text{average}}[\tau_a, N_0]$. In order to solve Problem 2, we then characterize sufficient regulatory conditions on the switching signals associated to these classes, such that the asymptotic synchronization of the switched triggered network (1) be guaranteed.

**A. Regulatory Conditions on $S_{\text{dwell}}[\tau_D]$ Class**

In this subsection we first discuss the switching between two topologies, we then characterize regulatory condition in terms of $\tau_D$, and triggering strategy (5), such that the asymptotic synchronization of the switched triggered network (1) be ensured.

**Theorem 5.1:** Consider dynamics (1), where $k \in \{1, 2\}$, and assume there exist matrices $P^k_i = P^k_{i \rightarrow 0} \in \mathbb{R}^{n \times n}$, $i \in \{2, 3, \ldots, N\}$, satisfying condition (4), as stated in proposition 3.4, for every $k \in \{1, 2\}$. Moreover, let $\delta \in (0, 1)$, $e(t) = x(t^k_i) - x(t)$, $\alpha_k = \max_{i \in \{2, 3, \ldots, N\}} \{ -\delta \lambda^2 P^k_i \}$, $\tau_D = \max_{i \in \{2, 3, \ldots, N\}} \{ P^k_i \}$, and $\{ t^k_i \}$ be the triggering time-sequence generated by (5), that is restated as follows:

$$\| \dot{V}^k e(t^k_i) \| = \frac{\delta}{\alpha_k} \| \dot{V}^k e x(t^k_i) \|, p \in \mathbb{N}, k \in \{1, 2\}. $$

Then, the switching signal, $\sigma : \mathbb{R}_{\geq 0} \rightarrow k$, where $\sigma(t) \in S_{\text{dwell}}[\tau_D]$, ensures the asymptotic synchronization of (1) under the following condition:

$$\tau_D > \frac{1}{2(1 - \delta)} \frac{\lambda_{\max}(P^2) P^k}{\lambda_{\max}(P^2) P^k} \max\{\Upsilon_1, \Upsilon_2\}, $$

with

$$\Upsilon_1 = \log \frac{\lambda_{\max}(P^1)^2 \lambda_{\max}(P^2)}{\lambda_{\min}(P^1)^2 \lambda_{\min}(P^2)}, $$

$$\Upsilon_2 = \log \frac{\lambda_{\max}(P^2)^2 \lambda_{\max}(P^1)}{\lambda_{\min}(P^2)^2 \lambda_{\min}(P^1)}. $$

The previous result has been established for the case of two topologies. Nonetheless, it can be extended to the case of switching amongst $m$ topologies. The following corollary elaborates on this point.

**Corollary 5.2:** Consider dynamics (1), where $k \in \{1, \ldots, m\}$, and assume that the assumptions stated in Theorem 5.1 hold. Moreover, let $\tau_D$ be the dwell time given by Algorithm 1. Then, the asymptotic synchronization of dynamics (1) is ensured under triggering strategy (8) and switching signal, $\sigma : \mathbb{R}_{\geq 0} \rightarrow k$, where $\sigma(t) \in S_{\text{dwell}}[\tau_D]$, for $\tau_D \geq \tau_D$.

**B. Regulatory Conditions on $S_{\text{average}}[\tau_a, N_0]$ Class**

In previous subsection, we discussed regulatory conditions on $S_{\text{dwell}}[\tau_D]$, in this subsection, we shall discuss switching signals, $\sigma(t) \in S_{\text{average}}[\tau_a, N_0]$. 

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Algorithm 1 $\tau_D$-Seeking

Input: $m$: number of topologies; matrices $P^i$ satisfying condition (4) for every $i \in \{1, \ldots, m\}$; $\lambda_{\text{max}}(P^i)$ for every $i \in \{1, \ldots, m\}$.

1: for $i = 1$ to $m - 1$ do
2: for $j = 1$ to $m - 1$ do
3: for $k = 1$ to $m$ do
4: Compute the possible dwell-times:
   \[
   \tau_D(i, j, k) = \frac{1}{2(1 - \delta)} \prod_{q=i}^{m} \lambda_{\text{max}}(P^q) \times \log \left( \frac{\lambda_{\text{max}}(P^k) \prod_{q=i}^{m} \lambda_{\text{max}}(P^q)}{\lambda_{\text{min}}(P^k) \prod_{q=i}^{m} \lambda_{\text{min}}(P^q)} \right).
   \]
5: end for
6: end for
7: end for

Output: $\tau_D = \max_{i,j,k} \{\tau_D(i, j, k)\}$.

For this class, the number of discontinuities of switching signal, $\sigma(t)$, over time-interval $(t, \tau)$, i.e., $N_\sigma(t, \tau)$, satisfies $N_\sigma(t, \tau) \leq N_0 + \frac{\lambda_{\text{max}}(\Gamma)}{\lambda_{\text{min}}(\Gamma)}$ with parameters $N_0$ and $\tau_\alpha$ be respectively average dwell time and chatter bound. The main goal of this subsection is to characterize sufficient conditions on these two parameters such that the asymptotic synchronization of the triggered network dynamics (1) is guaranteed.

Theorem 5.3: Consider dynamics (1), where $k \in \{1, \ldots, m\}$, and assume there exist matrices $P_k^i = P_k^{i\top} > 0 \in \mathbb{R}^{n \times n}$, $i \in \{2, 3, \ldots, N\}$, satisfying condition (4), as stated in Proposition 3.4, for every $k \in \{1, \ldots, m\}$. Moreover, let $\delta \in (0, 1)$, $\alpha_k = \max_{i \in \{2, 3, \ldots, N\}} \{-c_{\lambda_k}^i ||P_k^{i\top}\Gamma||\}$, $P_k^i = \text{diag}_{i \in \{2, 3, \ldots, N\}} \{P_k^i\}$, and $\{t_p^i\}$ be the switching time-sequence generated by (5), that is restated as follows:

\[
||\Phi_{k\top}^i e(t_p^i)\| = \frac{\delta_k}{\alpha_k} ||\Phi_{k\top}^i x(t_p^i)\|, \quad p \in \mathbb{N}, \quad k \in \{1, \ldots, m\}.
\]

Then, the switching signal, $\sigma: \mathbb{R}_+ \rightarrow k$, where $\sigma(t) \in S_{\text{average}}[\tau_\alpha, N_0]$, ensures the asymptotic synchronization of (1) with $N_0 \in \mathbb{N}$, and under the following condition:

\[
\tau_\alpha > \frac{\lambda_{\text{min}}}{2(1 - \delta)} \log \left( \frac{\lambda_{\text{max}}}{\lambda_{\text{min}}} \right),
\]

where,

\[
\tilde{\lambda}_{\text{min}} = \max_{k \in \{1, \ldots, m\}} \{\lambda_{\text{min}}(P^k)\},
\]

\[
\tilde{\lambda}_{\text{max}} = \max_{k \in \{1, \ldots, m\}} \{\lambda_{\text{max}}(P^k)\},
\]

\[
\lambda_{\text{min}} = \min_{k \in \{1, \ldots, m\}} \{\lambda_{\text{min}}(P^k)\}.
\]

In previous result, the asymptotic synchronization property is guaranteed under described conditions, where the decay rate of the synchronization cannot be designed. Indeed, an alteration in the triggered triggering strategy can help us characterize an exponential decay rate for the asymptotic synchronization.

Corollary 5.4: Consider dynamics (1), where $k \in \{1, \ldots, m\}$, and assume there exist matrices $P_k^i = P_k^i > 0 \in \mathbb{R}^{n \times n}$, $i \in \{2, 3, \ldots, N\}$, satisfying condition (4), as stated in Proposition 3.4, for every $k \in \{1, \ldots, m\}$. Moreover, let $\delta_1 > 0$, $\delta_2 \in (0, 1)$, such that $\delta_1 + \delta_2 \in (0, 1)$, $e(t) = x(t_p^i) - x(t)$, $\alpha_k = \max_{i \in \{2, 3, \ldots, N\}} \{-c_{\lambda_k}^i ||P_k^{i\top}\Gamma||\}$, $P_k^i = \text{diag}_{i \in \{2, 3, \ldots, N\}} \{P_k^i\}$, and $\{t_p^i\}$ be the switching time-sequence generated by (5), that is restated as follows:

\[
||\Phi_{k\top}^i e(t_p^i)\| = \frac{\delta_k}{\alpha_k} ||\Phi_{k\top}^i x(t_p^i)\|, \quad p \in \mathbb{N}, \quad k \in \{1, \ldots, m\}.
\]

Then, the switching signal, $\sigma: \mathbb{R}_+ \rightarrow k$, where $\sigma(t) \in S_{\text{average}}[\tau_\alpha, N_0]$, assures the exponential decay rate of: $\lambda^* = \frac{\lambda_{\text{max}}}{\lambda_{\text{min}}}$, with $N_0 \in \mathbb{N}$, and under the following condition:

\[
\tau_\alpha > \frac{\tilde{\lambda}_{\text{min}}}{2(1 - (\delta_1 + \delta_2))} \log \left( \frac{\tilde{\lambda}_{\text{max}}}{\lambda_{\text{min}}} \right),
\]

where the parameters $\tilde{\lambda}_{\text{min}}$, $\tilde{\lambda}_{\text{max}}$, and $\lambda_{\text{min}}$ are as stated in Equation (9) of Theorem 5.3.

VI. SIMULATIONS

In this section, we demonstrate the functionality of the regulatory conditions on $S_{\text{dwell}}[\tau_D]$ and $S_{\text{average}}[\tau_\alpha, N_0]$ classes of switching signals as discussed in Section V.

c) Network Description: In this set of simulations, we have considered network of $N = 5$, and $n = 2$.

Accordingly, matrices $H = \begin{bmatrix} 0 & -0.5 \\ 0.5 & 0 \end{bmatrix}$, and

$\Gamma = \begin{bmatrix} 0.25 & 0 \\ -1 & 0.25 \end{bmatrix}$, have been considered. We have then considered three potential topologies for the network, with the outer-coupling matrices, $A^1 = \begin{bmatrix} -3 & 1 & 1 & 0 & 1 \\ 1 & -3 & 1 & 1 & 0 \\ 1 & 1 & -4 & 1 & 1 \\ 0 & 1 & 1 & -3 & 1 \\ 1 & 0 & 1 & 1 & -3 \end{bmatrix}$,

$A^2 = \begin{bmatrix} -2 & 1 & 1 & 0 & 0 \\ 1 & -3 & 1 & 1 & 0 \\ 1 & 1 & -4 & 1 & 1 \\ 0 & 1 & 1 & -3 & 1 \\ 0 & 0 & 1 & 1 & -2 \end{bmatrix}$, and $A^3 = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix}$.

d) Simulations for $S_{\text{dwell}}[\tau_D]$: We shall consider the “worst-case switching scenario,” where every topology is active for $\tau_D$ seconds. Also, provided we have considered 3 topologies, the appropriate parameter $\tau_D$ is obtained recalling Algorithm 1. Let then $\delta = 0.5$, we
obtain $\tau_D = 9.8993$. We then conduct the simulations for the afore discussed switching scenario, where we consider consecutive time-intervals, $[n\tau_D, (n+1)\tau_D]$, $n \in \mathbb{N}_0$, where for $n = 3k$, $n = 3k + 1$, and $n = 3k + 2$, with $k \in \mathbb{N}_0$, the first, second, and third topology are active, respectively. Also, for every time-interval, $[n\tau_D, (n+1)\tau_D]$, the triggering strategy (8) is implemented. The results are shown in Figure 1, where asymptotic synchronization despite presence of switching and under triggering strategy is demonstrated.

e) Simulations for $S_{\text{average}}[\tau_a, N_0]$: In this set of simulations, we again consider the set of parameters introduced earlier. We then consider the time-intervals, $(t, \tau) = ((3n\tau_a, 3(n+1)\tau_a)$, with $n \in \mathbb{N}_0$, where the number of discontinuities is characterized as $N_\sigma(3n\tau_a, 3(n+1)\tau_a) \leq N_0 + 3$. We also consider $N_0 = 1$, which then yields $N_\sigma(3n\tau_a, 3(n+1)\tau_a) \leq 4$. Hence, we pick $N_\sigma(3n\tau_a, 3(n+1)\tau_a) = 3$, which infers that we consider the number of discontinuities over every $(3n\tau_a, 3(n+1)\tau_a)$ time-interval to be 3. In addition, we add that within every $(3n\tau_a, 3(n+1)\tau_a)$, we consider the subintervals, $(3n\tau_a, (3n+1)\tau_a)$, $((3n+1)\tau_a, (3n+2)\tau_a)$, and $((3n+2)\tau_a, 3(n+1)\tau_a)$ wherein the third, second, and first topologies are active, respectively. Let then $\delta = 0.5$, for the set of parameters stated previously, we obtain $\tau_a = 1.5350$. We then conduct our simulation along the lines of the afore discussed switching scenario, where for every time-interval, $[n\tau_a, (n+1)\tau_a]$, the triggering strategy (8) is implemented. The results are shown in figure 2, where asymptotic synchronization despite presence of switching and under triggering strategy is demonstrated.

VII. Conclusions and Future Work

In this paper, we have considered a specific class of synchronization problems with identical nodes. We have then recalled and characterized specific event-triggered rules to ensure the synchronization of this class of dynamics. We have then studied the robustness of this class of event-triggered synchronization dynamics in the face of various switching scenarios. More specifically, under arbitrary switching scenario, we characterized conditions on network topology to achieve asymptotic synchronization. At last, we have also characterized certain robustness conditions for two classes of constrained switching signals, namely uniform and average, under which asymptotic synchronization is ensured.

As future work, there are several venues to be yet explored. This includes, studying the similar type of robustness on switching signals for (i) distributed event-triggered rules—this latter would entail considering a slightly different type of dynamics for the network, for which the analyses presented in this paper would serve as an initial point, and (ii) other classes of synchronization problems, i.e., with different isolated node dynamics.

REFERENCES