An Intrinsic Robust PID Controller on Lie Groups

D.H.S. Maithripala and J. M. Berg

Abstract—This paper presents a PID controller for fully actuated, left-invariant, mechanical systems on a general Lie group. The class of problems solved includes tracking of a smoothly time-varying desired orientation for a rigid body with fully actuated attitude dynamics in two or three dimensions. If the reference velocity and unmodeled disturbance forces converge to constant values, then the closed-loop system will be almost-globally exponentially stable. The controller is robust to errors or variations in the inertial parameters and the actuator parameters. We explicitly construct the controller on the group of rigid body rotations and demonstrate its performance in quadrotor attitude tracking.

I. INTRODUCTION

In this paper we consider fully actuated, left invariant mechanical systems on a Lie group. These are in general highly non-linear systems. The use of linear tracking controllers for such systems will not perform well away from their designed operating point. Thus requiring multiple operating regions to be combined using schemes such as gain scheduling. These approaches may have difficulty when for example a vehicle maneuvers aggressively. Nonlinear controllers employing a single set of coordinates can greatly extend the operating region, but there are configuration spaces of interest, such as the space of rigid body rotations $\text{SO}(3)$, that are not globally diffeomorphic to $\mathbb{R}^n$. It has been shown that there exists no continuous state feedback that makes a given configuration globally asymptotically stable if the underlying space is not diffeomorphic to $\mathbb{R}^n$ [1]. The consequences may be seen in the rigid-body attitude control problem, where controllers written using minimal coordinates, such as Euler angles, fail at singular points, while controllers written using quaternions, suffer from ambiguity due to nonuniqueness, which can lead to poor performance [2]. These issues are addressed by the geometric approach, where the controller is derived independently of any choice of coordinates. While specific implementations may require the coordinates to be specified, the system will transition seamlessly across coordinate patches, and coordinates can be chosen to avoid ambiguity. These are very desirable qualities for reliable and safe operation of aircraft and spacecraft. Highly maneuverable vehicles such as small quadrotors require global or almost global stabilization properties to allow large and rapid changes in orientation without loss of control.

A simple mechanical system on a configuration space is defined by the kinetic energy and the forces acting on the system. The configuration space is generally a Riemannian manifold with the metric derived from the kinetic energy. The covariant derivative associated with the metric gives rise to the so-called intrinsic acceleration, which does lie in the tangent space to the manifold and permits an intrinsic version of Newton’s second law. This covariant derivative is the basis of our intrinsic PID control formulation.

Intrinsic PD controllers found in the literature are constructed using an appropriate scalar function that intrinsically captures the notion of tracking error. Such an error function is required to be a polar Morse function, that is, a Morse function that attains a unique minimum. The intrinsically defined gradient of the error function plays the role of the proportional part of the controller while an intrinsic velocity error feedback term plays the role of the derivative term. It has long been known that polar Morse functions are guaranteed to exist on any smooth manifold without a boundary [3]. The seminal work of [4] extended this result to manifolds with boundary. In intrinsic PD control, the polar Morse error function plays the role of potential energy and naturally gives rise to a Lyapunov function that is nonincreasing along any trajectory of the system, and a geometric version of the LaSalle invariance principle ensures that the system must converge to a forward invariant manifold. However, since any smooth vector field on a compact manifold with an asymptotically stable equilibrium must necessarily have at least one unstable equilibrium, the best that can be obtained with smooth feedback control is almost-global stability. Intrinsic PD controllers providing almost-global stability may be found, for example, in [4], [5], [6], [7], [8].

Intrinsic PD control guarantees global boundedness of the tracking error in the presence of constant parameter errors and disturbances. However, as for the linear PD controller, the tracking error in such cases will not converge to zero. As in the linear case, the error can be made arbitrarily small by picking the PD gains sufficiently large, however doing so has undesirable consequences, such as the need for large actuators, the possibility of saturation, and amplification of noise. Therefore a natural extension of the intrinsic PD controller to ensure zero tracking error in the presence of constant disturbances would be to incorporate an integrator term [9], [10]. Such PID controllers have been proposed for the special case of attitude tracking in [11], [12], [13]. However these are all based on local coordinates or quaternion representations, and therefore suffer poor global performance from either singularities or ambiguity. Furthermore, they are not intrinsic, and therefore cannot be generalized to a broader class of systems.

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The PID formulation proposed in [14], [15], [16] for attitude tracking in SO(3) is coordinate-free, and the controller provides excellent global convergence properties in the presence of constant disturbances. However, the robustness of the scheme to uncertainties in inertial parameters and the actuation force models is not considered. Such uncertainties typically have an effect similar to that of a time-varying disturbance. It is not clear if the controller reported in [14], [15], [16] preserves its convergence properties in the presence of such disturbances. The PID formulation proposed in [14], [15], [16] again uses the intrinsic gradient of the error function to provide the proportional term, while the error system velocities provide the derivative term. A differential equation defines the integral term by setting the time derivative of the integral error to be the intrinsic gradient of the error function plus a velocity term. The velocity term in the integrator is a clear distinction from its linear counterpart. Furthermore, the time derivative used in this definition is not the covariant derivative. As a result, the integrator is not intrinsic, because the form of the integrator depends on the coordinates of the Lie algebra so(3) of the Lie group SO(3). The controller is developed using the canonical coordinates of so(3). Since the tangent bundle to any Lie group can be expressed as the global product of the Lie group and its Lie algebra, and the Lie algebra is isomorphic to some \( \mathbb{R}^n \), the general extension of this controller to Lie groups is possible. However because the integral term is not intrinsic, further generalization to Riemannian manifolds is not clear.

In this paper we provide a truly intrinsic PID controller, formulated in the setting of a simple mechanical system on a Lie group. The controller is invariant under coordinate transformations of the tangent bundle. It is expressed using the Riemannian structure of the configuration manifold. Our intrinsic PID control is an exact analog to linear PID control, and reduces to the standard formulation when applied to mechanical systems in Euclidean space. A polar Morse function is chosen as the error function. The proportional control is given by the intrinsic gradient of the error function, the derivative term is given by feedback of the intrinsically defined velocity of the error dynamics, and the covariant derivative of the integral term is the intrinsic gradient of the error function. In this way, the integral term is made to lie in the tangent bundle. The approach is directly extensible to Riemannian manifolds, but for clarity we present only the Lie group version here.

In this paper, the intrinsic PID controller is developed for a fully actuated mechanical system on a general Lie group with left invariant kinetic energy, for which rigid-body attitude tracking is a special case. We show that it locally exponentially stabilizes any twice differentiable desired reference, for all initial conditions except a set of initial conditions of measure zero, in the presence of bounded disturbances and bounded parametric uncertainties in inertia properties and control force models if the disturbances and the references velocities approach a constant value. We also show that the trajectories are guaranteed to be bounded in the presence of parametric uncertainty for a large set of initial conditions for general bounded disturbances and bounded references. If the Lie group is compact then the trajectories are globally bounded. We also show using the separation principle proved in [7] that the implementation of the controller in conjunction with any almost globally convergent configuration and velocity estimator such as the one proposed in [17] for rigid body motion in SO(3) preserves the convergent properties of the controller. To the best of our knowledge it is the first time that such a general result has been proven. The proof relies on the Riemannian structure of the Lie group, and the existence of a polar Morse function and hence the results are applicable for any fully actuated mechanical system on a Riemannian manifold. These results will be presented in a subsequent work. A special case of these results applicable to only configuration stabilization on SO(3) was presented in [18].

In section II we review basic concepts related to simple mechanical systems on general manifolds and Lie groups. Section III contains the main contribution—the statement of an intrinsic PID controller for fully actuated mechanical systems on Lie groups. The proof of this result follows the proof, sketched out in [18], for the special case of constant set point tracking on SO(3). The details of the proof become slightly more involved due to the consideration of time varying references. Thus due to a lack of space the proofs will be omitted from here and will be included in an expanded future version where we will further extend these results to the tracking of classes of references [19]. Section III also presents a separation principle for the controller. We apply the results developed for general Lie groups to the special case of the compact Lie group of rigid body rotations SO(3) in Section IV. The controllers are developed without parameterization of SO(3) or so(3). In Section V we examine the performance of the controller for attitude tracking and attitude stabilization of a simulated quadrotor.

II. INTRINSIC MECHANICAL SYSTEMS

Geometric mechanics is the natural setting for mechanical systems since the configuration spaces are typically non-Euclidean manifolds \( \mathcal{Q} \). We denote a particular configuration by \( q \in \mathcal{Q} \), and a system trajectory by \( q(t) \). The velocity of \( q(t) \) is well defined, and is denoted by \( \dot{q}(t) \equiv v(t) \). The velocity \( \dot{q}(t) \) is a vector in the tangent space of the configuration manifold at \( q \), written \( T_q \mathcal{Q} \). The configuration manifold together with the tangent space at each point is called the tangent bundle, written \( T \mathcal{Q} \). The dual space \( T^* \mathcal{Q} \) is called the co-tangent bundle. A mechanical system on a configuration space \( \mathcal{Q} \) is defined by the kinetic energy of the system, and the forces acting on the system. Being equal to the rate of change of momentum, forces are intrinsically covectors and hence elements of the cotangent bundle \( T^* \mathcal{Q} \). The kinetic energy, KE typically allows one to define an inner product \( \langle \cdot, \cdot \rangle : T_q \mathcal{Q} \times T_q \mathcal{Q} \to \mathbb{R} \) on each of the tangent spaces \( T_q \mathcal{Q} \) by the relationship \( KE = \frac{\langle \dot{q}(t), \dot{q}(t) \rangle}{2} \). This defines a Riemannian metric on \( \mathcal{Q} \). The metric also induces an isomorphism \( \pi : T_q \mathcal{Q} \to T_q^* \mathcal{Q} \) by the relationship...
\( \dot{v}_{q}(u_{q}) \triangleq \langle \langle v_{q}, u_{q} \rangle \rangle \) for all \( v_{q}, u_{q} \in T_{q}Q \). The mapping of the time derivative of momentum into the tangent space is called the intrinsic acceleration. The coordinate-independent definition of intrinsic acceleration is given in terms of a uniquely defined covariant derivative or connection, called the Levi-Civita connection that is associated with the kinetic energy metric. The covariant derivative of vector field \( Y \) along vector field \( X \) is written \( \nabla_{X}Y \). The covariant derivative \( \nabla_{X}Y \) in this paper always refers to the Levi-Civita connection associated with the kinetic energy metric.

The intrinsic acceleration for velocity \( v(t) \) is the covariant derivative of \( v(t) \) with respect to itself, that is, \( \nabla_{v(t)}v(t) \).

Let \( F = \mathbb{I}^{-1}f \in TQ \) be the vector version of the force \( f \in T^*Q \). We assume the vector version of the force to be of the form \( F = F^{u} + F^{c} + \Delta d \) where \( F^{c} \) denotes the modelled external physical constraint forces acting on the system such as gravity, drag, frictional, or no-slip type forces and \( F^{u} \) denotes the control moments. Here \( \Delta d \) denotes disturbances and un-modelled forces. Using the covariant derivative the Newton equations governing the system are given by \( \nabla_{v(t)}v(t) = F^{u} + F^{c} + \Delta d \). For simplicity in the following sections we will neglect the constraint force \( \mathbb{I}F^{c} \). The effect of these forces can be easily accounted for by incorporated a term in the controller to cancel its effect where the errors due to the uncertainties of the knowledge of \( F^{c} \) can be included as part of \( \Delta d \).

For mechanical systems where the configuration space \( Q \) is a Lie group and the kinetic energy metric is left invariant the mechanical system representation becomes simpler. Furthermore the Lie group structure allows one to define distance functions and error functions intrinsically without resorting to co-ordinates. In what follows we concentrate on fully actuated mechanical systems on a Lie Group \( G \). An element in \( G \) will be referred to as \( g \in G \). The identity element of \( G \) will be denoted by \( e \). Let \( \mathcal{G} \triangleq T_{e}G \) be the Lie algebra of the Lie group \( G \). It follows that \( \dot{g} = g \cdot \zeta \) where \( \zeta \in \mathcal{G} \) and \( g \cdot \zeta \) means the left translation and \( \zeta \cdot g \) means the right translation of \( \zeta \in G \in T_{g}G \) to \( T_{g}G \). The adjoint action \( Ad: \mathcal{G} \times G \rightarrow G \) is defined as \( Ad_{g}\zeta = g \cdot \zeta \cdot g^{-1} \). The associated \( ad: \mathcal{G} \times \mathcal{G} \rightarrow \mathcal{G} \) map is defined to be \( ad\zeta \eta \equiv [\zeta, \eta] \) where \( [\cdot, \cdot] \) is the Lie bracket on \( \mathcal{G} \). The dual of this map is denoted by \( ad^{*} \). We will only consider mechanical systems with a left invariant metric. In this case the pull-back of the map \( l \) to \( T_{e}G = G \) is a constant and is usually referred to as the inertia tensor. The Levi-Civita connection associated with a left invariant metric takes the form \( \nabla_{g}g \cdot \eta = g \cdot \nabla\zeta \eta \) where

\[
\nabla\zeta \eta = \eta(\zeta) + \frac{1}{2} \left( ad\zeta \eta - \eta^{-1} (ad\zeta^{*}\eta + ad\eta^{*}\zeta) \right),
\]

for \( \zeta, \eta \in \mathcal{G} \). Note that \( \nabla\zeta \zeta = \dot{\zeta} - \eta^{-1} ad^{*}\eta \zeta\).

Then Newton equations are intrinsically represented by

\[
\dot{g} = g \cdot \zeta, \quad (1)
\]

\[
\nabla\zeta \zeta = F^{u} + F^{c} + \Delta d. \quad (2)
\]

### III. Almost Globally Stable PID Control On Lie Groups

Given a desired twice differentiable configuration reference \( g_{r}(t) \in G \) we are interested in finding a control that will ensure \( \lim_{t \rightarrow \infty} g(t) = g_{r}(t) \) for a very large set of initial conditions in the presence of un-modeled forcing and parametric uncertainty. The Lie group structure allows one to define the intrinsic configuration error \( E = g_{r}^{-1}g \) [4], [5], [6], [7]. The the velocity error \( \zeta_{E} \) by the relationship

\( E = E \cdot \zeta_{E} \),

with \( \dot{g}_{r} = g_{r} \cdot \zeta_{r} \).

We recall that \( \Delta d_{E^{-1}}g_{r} = E^{-1} \cdot \zeta_{r} \cdot E \).

Let \( f: G \rightarrow \mathbb{R} \) be a polar Morse function on \( G \). Polar Morse functions exist on any smooth manifold without a boundary [3] and on compact manifolds with a boundary [4]. The function \( f(E) \) is referred to as the error function. It helps quantify the configuration error. Larger the error larger the value of the error function is. Let \( \eta_{E} \triangleq E^{-1} \cdot \text{grad} f(E) \in T_{E}G \). It can be shown that the intrinsic nonlinear PD-controller

\[
F^{u} = - k_{p} \eta_{E} - k_{d} \zeta_{E} + F_{r}(E, \zeta_{E}, \zeta_{r}) (3)
\]

where

\[
F_{r} = \nabla\zeta \Delta d_{E^{-1}} \zeta_{r} + \nabla \Delta d_{E^{-1}} \zeta_{r} - \nabla \Delta d_{E^{-1}} \zeta_{r} \Delta d_{E^{-1}} \zeta_{r}. \quad (4)
\]

almost globally and locally exponentially stabilizes the reference configuration \( g_{r}(t) \) in the absence of uncertainties [4], [5], [6]. As in linear PD control the proportional part of the controller assigns the error function \( f(E) \) as the potential energy of the error dynamics while the derivative control adds damping to the system.

The proof of the convergence properties also shows that in the presence of bounded constant disturbances and modelling errors in the parameters and control moments the trajectories can be made to converge almost globally to an attitude close to \( g_{r}(t) \) if the gains \( k_{p}, k_{d} \) are large enough. Picking \( k_{p}, k_{d} \) to be large has the undesirable effect of magnifying measurement noise. Furthermore in applications such as quad rotor stabilization constant errors in the stabilized attitude will cause the device to drift in 3D space. To overcome this problem we propose the use of an intrinsic PID controller.

Let \( \zeta_{l} \in \mathcal{G} \). The intrinsic PID controller we propose is

\[
\nabla_{g_{l}} \zeta_{l} = \eta_{E}, \quad (5)
\]

\[
F^{u} = - k_{l} \zeta_{l} - k_{p} \eta_{E} - k_{d} \zeta_{E} + F_{r}, \quad (6)
\]

The system (1), (2) together with the integrator (5) defines a dynamic system on \( \mathcal{X} \triangleq \mathcal{G} \times G \times \mathcal{G} \).

**Remark 1:** Observe that the intrinsic controller has been expressed using the Riemannian structure of the configuration space and thus is extendable in a straightforward manner to fully actuated mechanical systems on a general Riemannian manifold. We will see that the key to the convergence properties of the controller depend on the existence and use of a proper error function. Polar Morse functions are ideal candidates for these error functions [4]. The properties of Morse functions are uniquely determined by the underlying topology of the space [3] and thus the global convergence properties of the general PID controller will also be governed by the topology of the underlying configuration space.
The closed loop error dynamics of the system evolve on $G \times G \times G$ and is given by
\begin{equation}
\dot{E} = E \cdot \zeta_E, \quad (7)
\end{equation}
\begin{equation}
\nabla \zeta_E \zeta_E = -k_I \zeta I - k_p \eta E - k_d \zeta E + \Delta_d + \Delta_e, \quad (8)
\end{equation}
\begin{equation}
\nabla \zeta E \zeta I = \eta E, \quad (9)
\end{equation}
where the parameter error term $\Delta_e$ is given by $\Delta_e \triangleq \epsilon(-k_I \zeta I - k_p \eta E - k_d \zeta E + F_r)$. Here $\epsilon$ is a linear operator, $\epsilon : G \rightarrow G$ that depends on $(I-\bar{I}_0 - I_{nxn})$ and the actuator model uncertainties where $\bar{I}$ is the actual inertia tensor and $I_0$ is the nominal inertia tensor with $n = \dim(G)$.

Consider a possibly large compact subset $\mathcal{X} \subset G \times G \times G$. Let $\lambda, \mu > 0$ be such that $\|\zeta E \|_2^2 \leq \lambda \|\eta E\|_2^2$, $\|\zeta I\|_2^2 \leq \mu \|\zeta I\|_2^2$, for $(E, \zeta, \eta) \in \mathcal{X}$. The existence of such $\lambda, \mu$ is guaranteed on any compact set. The first of the above two conditions is implied by the fact that $f(E)$ is quadratically bounded from below at the global minimum and the boundedness of the gradient while the second one is implied by the boundedness of the Hessian. Let $k_p, k_I, k_d > 0$ be such that
\begin{equation}
0 < k_I < k_d (1 - \delta^2), \quad (10)
\end{equation}
\begin{equation}
k_p > \max \left\{ \frac{\lambda k_d^2}{2k_d^2} \left( 1 + \sqrt{1 + \frac{4k_d^2(k_d^2 + 4k_d^2k_d^2)}{\lambda k_d^2}} \right) \right\}, \quad (11)
\end{equation}
where $0 < \kappa < 2/\mu$ and $\delta = |(\kappa \mu - 1)| < 1$.

The following two theorems show that the intrinsic PID controller robustly tracks a given desired twice differentiable reference configuration locally exponentially. Due to a lack of space the proofs will be omitted from here and will be included in an expanded future version where we will further extend these results to output tracking using state feedback [19].

**Theorem 1:** Consider the mechanical system (1)–(2) with bounded disturbance $\Delta_d(t)$. The intrinsic PID controller given by (5)–(6) with gains chosen such that (10)–(11) are satisfied, ensure that the trajectories of the closed loop error system (7)–(9) remain bounded for any initial condition that belongs to a large set, for any bounded twice differentiable reference $g_r(t)$, in the presence of bounded parametric uncertainty. The intrinsic tracking error and the integrator state can be made arbitrarily small by picking the gains to be sufficiently large. The boundedness holds globally if the Lie group $G$ is compact.

**Theorem 2:** For the mechanical system (1)–(2) the intrinsic PID controller given by (5)–(6) with gains chosen such that (10)–(11) are satisfied, ensure that $\lim_{t \rightarrow \infty} \zeta r(t) = g_r(t)$ almost globally and locally exponentially in the presence of parametric uncertainties if the once differentiable reference velocity $\zeta r(t) = g_r^{-1} g_r(t)$ and disturbances $\Delta_d(t)$ asymptotically approach a constant.

The implementation of the PID control (5)–(6) typically requires an observer or complementary filter that estimates $g$ and $\zeta$. Let $(g_r(t), \zeta r(t))$ be estimates of $(g(t), \zeta(t))$ such that $\lim_{t \rightarrow \infty} (g_r(t), \zeta r(t)) = (g(t), \zeta(t))$ exponentially for all initial conditions that belong to $(g(0), \zeta(0)) \in \mathcal{X}_0 \subset G \times G$. Define $\tilde{E} \triangleq g_r^{-1} g_r$ and $\tilde{\eta} E$ to be the gradient of $f(E)$. Let $\zeta E \triangleq \zeta - \zeta d - \zeta r$. With these estimates the PID controller takes the form
\begin{equation}
\nabla \zeta E \zeta I = \tilde{\eta} E, \quad (12)
\end{equation}
\begin{equation}
F' = -k_I \zeta I - k_p \tilde{\eta} E - k_d \zeta E + F_r(\tilde{E}, \tilde{\zeta} E, \zeta r). \quad (13)
\end{equation}

In the case of compact Lie groups using the separation principle proved in [7] one can show that this controller preserves its convergent properties if $g(t)$ and $\zeta(t)$ are estimated by an observer or a complementary filter with locally exponential convergence. The key to the proof is the global boundedness of trajectories reported in Theorem 1 for bounded disturbances. The proof follows exactly the proof presented in [7] and therefore will be omitted from this paper. We state this in the following theorem:

**Theorem 3:** For the mechanical system (1)–(2) on a compact Lie group let $(g_r(t), \zeta r(t))$ be estimates of $(g(t), \zeta(t))$ such that $\lim_{t \rightarrow \infty} (g_r(t), \zeta r(t)) = (g(t), \zeta(t))$ exponentially for all initial conditions that belong to $(g(0), \zeta(0)) \in \mathcal{X}_0 \subset G \times G$, then the intrinsic PID controller given by (12)–(13) with gains chosen such that (10)–(11) are satisfied ensure that $\lim_{t \rightarrow \infty} g_r(t) = g_r(t)$ almost globally in $\mathcal{X}_0$ and locally exponentially, in the presence of parametric uncertainties if the once differentiable velocity reference $\zeta r(t) = g_r^{-1} g_r(t)$ and disturbance $\Delta_d(t)$ asymptotically approach a constant.

**IV. Example on SO(3)**

In this section we apply the results of the previous section to fully actuated mechanical systems on $SO(3)$. The group $SO(3)$ being compact, the resulting PID controller is almost globally and locally exponentially stable with convergence guaranteed for all initial conditions except three special initial conditions that correspond to three unstable equilibria of the closed loop system. We refer the reader to [18] for an extended treatment of this example for configuration stabilization.

Let $R \in SO(3)$ be the configuration, $\Omega = [\Omega_1 \ \Omega_2 \ \Omega_3]^T \in \mathbb{R}^3$ be the body angular velocity and $\bar{I}$ be the inertia tensor of the body. Denote by $\tau = I(T'' + \Delta_d)$ the moments acting on the body expressed in the body frame where $T'' = \Omega''$ is the control moment and $\Delta_r = \Delta_d$ represents un-modeled moments. The Newton’s equations (1), (2) in this case are the Euler’s rigid body equations $\dot{R} = R \tilde{\Omega}_r$, $\nabla \tilde{\Omega}_r \Omega = T'' + \Delta_d$, where $\nabla \tilde{\Omega}_r \Omega = \tilde{\Omega} - \tilde{\Omega}^{-1}(\tilde{\Omega} \times \tilde{\Omega})$ and the $\tilde{\Omega}$ denotes the skew symmetric matrix version of $\Omega$ given by
\[
\tilde{\Omega} = \begin{bmatrix}
0 & -\Omega_3 & \Omega_2 \\
\Omega_3 & 0 & -\Omega_1 \\
-\Omega_2 & \Omega_1 & 0
\end{bmatrix}.
\]

We are interested in tracking a desired twice differentiable reference configuration $R_r(t) \in SO(3)$. The intrinsic tracking error is $E(t) \triangleq R_r^T(t) R(t)$. Let $f : SO(3) \rightarrow \mathbb{R}$
be the polar Morse function that is explicitly given by $f(E) = \text{trace}(I_{3\times3} - E)$. It can be shown that $f(E)$ has four critical points at $\bar{E}_0 = I_{3\times3}$, $\bar{E}_1 = \text{diag}\{1, -1, -1\}$, $\bar{E}_2 = \text{diag}\{-1, 1, -1\}$, $\bar{E}_3 = \text{diag}\{-1, -1, 1\}$. The first one is a global minimum while the others give rise to the maximum value of the function $[4]$. Furthermore it also shows that $\lambda = 2\lambda_{\text{max}}(I) / \lambda_{\text{min}}(I)^2$ and $\mu = 2(\lambda_{\text{max}}(I) + \lambda_{\text{min}}(I))$. Let $\Omega_E = \Omega - A_{\text{d}E} - \Omega = -E^T \Omega_e$ where $\Omega_e = R^T \dot{R}_e$. In this case the intrinsic PID-controller (5)–(6) takes the form
\begin{equation}
\nabla_{\Omega_E} \Omega_E = \eta_E, \tag{14}
\end{equation}
\begin{equation}
T^\alpha = -k_p \eta_E - k_d \Omega_E - k_I \Omega_I + F_r. \tag{15}
\end{equation}
where $\Omega_I \in \mathbb{R}^3$. It is clear that, for disturbances $\Delta_d$ and references $R_e$ such that $\Delta_d$ and $\Omega_e$ approach a constant values $\bar{\Delta}_d$ and $\bar{\Omega}_e$, respectively, the equilibria of the error system (7)–(9) are of the form $(\bar{E}_i, 0, (\epsilon + I_{n\times n})^{-1}(\bar{\Delta}_d - \epsilon \bar{E}_i^T \bar{\Omega}_e / (\bar{\Omega}_e \times \bar{\Omega}_e))) / k_I$, for $i = 0, 1, 2, 3$. Thus from theorem-2 it can be shown that the equilibria corresponding to $i = 1, 2, 3$ are unstable and $(I_{3\times3}, 0, (\epsilon + I_{n\times n})^{-1}(\Delta_d - \epsilon \bar{E}_i^T \bar{\Omega}_e / (\bar{\Omega}_e \times \bar{\Omega}_e))) / k_I$ is an almost globally and locally exponentially stable equilibrium of the closed loop (1), (2), and (5) in the presence of bounded uncertainties in inertia properties, and the control moment models. Furthermore Theorem-1 ensures that the trajectories remain globally bounded for any bounded non constant disturbances $\Delta_d(t)$ and velocity references $\bar{\gamma}_r(t)$.

We assume that the existence of an observer such as the one proposed in [17] to provide an estimate $(R_o(t), \Omega_o(t))$ that converges almost globally and locally exponentially to the actual values $(R(t), \Omega(t))$. For details of the explicit implementation of this observer we refer the reader to [17]. Let $\bar{E} = \bar{R}^T \bar{R}_o$, $\bar{\Omega}_E = (\bar{\Omega}_o - \bar{E} \bar{\Omega}_r)$ and $\bar{\eta}_E = \bar{R}^T \bar{R}_o - \bar{R}_o^T \bar{R}_r$. Then from Theorem-3 it follows that the controller
\begin{equation}
\bar{\Omega}_I = -\frac{1}{2} (\bar{\Omega}_o \times \bar{\Omega}_I - \bar{I}^{-1}(\bar{I} \bar{\Omega}_o \times \bar{\Omega}_o) + \bar{\eta}_E), \tag{16}
\end{equation}
\begin{equation}
T^u = -k_p \bar{\eta}_E - k_d \bar{\Omega}_E - k_I \bar{\Omega}_I + F_r(\bar{E}, \bar{\Omega}_E, \bar{\Omega}_r). \tag{17}
\end{equation}
ensures that $\lim_{t \to \infty} R(t) = R_e(t)$ exponentially for all initial conditions except the three initial conditions that correspond to the unstable equilibria $(E_i, 0, (\epsilon + I_{n\times n})^{-1}(\Delta_d - \epsilon E_i^T \bar{\Omega}_e / (\bar{\Omega}_e \times \bar{\Omega}_e))) / k_I$, for $i = 1, 2, 3$.

V. SIMULATION RESULT FOR 3D QUAD-ROTOR ATTITUDE TRACKING

In this section we provide simulation results that demonstrate the effectiveness of the controller (16)–(17) where the estimated values $(R_o, \bar{\gamma}_r)$ are obtained by implementing the almost globally convergent complementary filter proposed in [17].

In hover mode, if one neglects the effect of rotor flapping the nominal body moments generated by the rotors can be expressed as $[20]$
\begin{equation}
\tau^u = \begin{bmatrix}
0 & Lc_l & 0 & -Lc_l \\
Lc_l & 0 & Lc_l & 0 \\
-c_d & c_d & -c_d & c_d
\end{bmatrix}
\begin{bmatrix}
\omega_1^2 \\
\omega_2^2 \\
\omega_3^2 \\
\omega_4^2
\end{bmatrix} \tag{18}
\end{equation}
and the total thrust force as
\begin{equation}
f^u = c_l(\omega_1^2 + \omega_2^2 + \omega_3^2 + \omega_4^2) \tag{19}
\end{equation}
where $L$ is the distance from the center of mass of the quadrotor to the center of the rotor, $c_l$ is the coefficient of lift acting on a rotor, $c_d$ is the co-efficient of drag acting on a rotor, and $\omega_i$ is the angular velocity of the $i^{th}$ rotor.

We let the nominal inertia parameters used in the computation of the control thrust force $f^u = M_0 F^u$ and moments $\tau^u = \bar{I}_0 T^u$ to be $M_0 = 0.85$ kg, $\bar{I}_0 = \text{diag}(0.004, 0.004, 0.007)$ kg m$^2$ respectively and $M = 0.95$ kg, $\bar{I} = \text{diag}(0.005, 0.005, 0.008)$ kg m$^2$ be the actual inertial parameters in the simulation. The actual actuator parameters $c_l, c_d$ were taken to be 10% different from the nominal values. We let $F^u = 1.5g$.

From the nominal inertial properties we have $\lambda = 875, \mu = 1375$. We let $\kappa = 1/\mu$ and hence $\delta = 0$. Thus we have that $k_p = 1.5, k_I = 15, k_d = 30$ satisfy the conditions (10)–(11).

Simulations were carried out at a step size of $h = 1$ ms using a Runge-Kutta numerical differentiation algorithm based on the MATLAB ODE45 function. For the integration of the attitude dynamics the attitude was parameterized by the unit-quaternions. In the controller calculations the quaternions were always converted back to the corresponding rotation matrix and thus avoiding any ambiguity in the 2 to 1 representation of SO(3) using the unit quaternions. To represent realistic conditions the controller was only updated every 25 ms. The IMU measurements were assumed to be corrupted by white noise. We also assumed that the center of mass is off-set by $X = [1, 1, 1]$ cm and is unknown. The center of mass off-set gives rise to an un-modeled time varying moment $\Delta_d = -g(\bar{X} \times R^T(t)e_3)$. In all simulations the initial conditions correspond to an upside down configuration. We note that this does not correspond to one of the undesirable equilibria of the closed loop due to the presence of uncertainties.

Figure-1 shows the tracking error while Figure-2 shows the corresponding unit-quaternion corresponding to the simulated attitude thus demonstrating the excellent robustness properties of the PID (16)–(17). Finally Figure-3 demonstrates the robust attitude stabilization properties of the proposed controller. Further details of the application of this PID controller to attitude stabilization can be found in [18].
This paper presents an invariant PID controller for fully actuated simple mechanical systems on general Lie groups. The controller is an exact analog of the classical PID controller for linear systems. The controller is capable of ensuring locally exponential configuration tracking for a large class of initial conditions in the presence of parametric uncertainty and disturbance moments that tend to a constant for configuration references that tend to a constant velocity. If the configuration space is compact then the convergence is guaranteed almost globally. For general bounded disturbances the tracking error is guaranteed to be bounded for a large set of initial conditions. Once again the boundedness is global if the configuration space is compact. We also show that a separation principle holds and the controller can be implemented in conjunction with any almost globally/globally convergent state observer or complementary filter. The controller is derived and the proofs are based on the Riemannian structure of the configuration space and thus the results are easily extendable to general mechanical systems on Riemannian manifolds. The controller is explicitly derived for the special case of rigid body attitude tracking and is demonstrated using simulations for quad-rotor attitude tracking.

VI. CONCLUSION

This paper presents an invariant PID controller for fully actuated simple mechanical systems on general Lie groups. The controller is an exact analog of the classical PID controller for linear systems. The controller is capable of ensuring locally exponential configuration tracking for a large class of initial conditions in the presence of parametric uncertainty and disturbance moments that tend to a constant for configuration references that tend to a constant velocity. If the configuration space is compact then the convergence is guaranteed almost globally. For general bounded disturbances the tracking error is guaranteed to be bounded for a large set of initial conditions. Once again the boundedness is global if the configuration space is compact. We also show that a separation principle holds and the controller can be implemented in conjunction with any almost globally/globally convergent state observer or complementary filter. The controller is derived and the proofs are based on the Riemannian structure of the configuration space and thus the results are easily extendable to general mechanical systems on Riemannian manifolds. The controller is explicitly derived for the special case of rigid body attitude tracking and is demonstrated using simulations for quad-rotor attitude tracking.

REFERENCES


